Plane Symmetric Higher Dimensional Vacuum Solutions in *f*(R) Gravity Theory

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Abstract: In this paper, we have studied the higher-dimensional plane symmetric vacuum solutions in *f*(R) gravity, where R is the Ricci scalar of the spacetime. we have assumed a constant scalar curvature, solved the field equations in higher dimensions in *f*(R) gravity, and we obtained exactly the same as the well-known Taub's like metric and anti-de Sitter spacetime metric respectively.

Keywords: *f*(R) gravity, higher dimensional, plane symmetric solution.

1.INTRODUCTION: -

The General Theory of Relativity (GR) is a remarkable and successful theory of gravitation that has greatly influenced modern physics. Since the 1950s, it has resolved numerous scientific questions. However, there remain challenging issues that GR cannot fully address, particularly the mysteries of dark energy and the accelerated expansion of the universe. Observations indicate that a vast majority of the universe's energy content, approximately 96 percent, is composed of unknowable substances—dark energy (about 76 percent) and dark matter (around 20 percent) [1]. In recent years, scientists working within the framework of GR have turned their attention to the cosmological constant (Λ) as a potential explanation for dark energy. Additionally, they've encountered the concept of singularities, which are common in certain cosmological models as per the singularity theorem. Exploring alternative theories has become necessary to address these challenges. One such theory gaining attention is *f*(R) gravity, which involves a function of the Ricci scalar R. Researchers are investigating this theory as a possible solution to the current cosmic acceleration, aiming to account for it without the need for dark energy [2-5]. This accelerated expansion of the universe is supported by recent cosmic observations [6-9]. Direct evidence for this cosmic acceleration comes from observations of supernovae [10], while indirect confirmation comes from sources like the cosmic microwave background [11] and the large-scale structure of the universe [12]. Basically, one of the most interesting mysteries in modern cosmology is how to explain why the universe is expanding faster. Alternative theories such as *f*(R) gravity offer fresh perspectives and potential solutions to these profound cosmological problems [13].

The *f*(R) actions were initially examined by H. Weyl in 1919 [14] and A. Eddington in 1922 [15]. Hans Adolph Buchdahl went on to study these phenomena in depth in 1970, focusing on non-singular oscillating cosmologies in particular [16]. Much study has been conducted on the theory of *f*(R) gravity, including studies of plane-symmetric solutions by Sharif and Shamir [17]. Amendola et al. made significant contributions by describing the conditions under which dark energy *f*(R) models are cosmologically viable [18]. Y. Aditya, R.L. Naidu, and D.R.K. Reddy also examined the non-static plane-symmetric perfect fluid cosmological model in the framework of *f*(R) gravity theory. They investigated the physical significance of cosmological parameters in the context of cosmic evolution as well as evaluating them for their model [19]. Several aspects of the *f*(R) theory of gravity have lately been examined [20-24], shedding light on a number of topics. Many researchers have concentrated on the exploration of plane- symmetric spacetime, and various approaches have been pursued. Plane wave solutions have been recognized and investigated by researchers such as Taub [25], Bondi [26] and Pirani-Robison [27]. Several authors [28-32] have examined multi-dimensional cosmological models within the context of both General Relativity and other modified theories of gravitation to advance our understanding of the cosmos.

The primary goal of this paper is to investigate exact solutions for higher-dimensional, plane-symmetric vacuum configurations that are static in a specific form of five-dimensional plane-symmetric spacetime with the coefficient of dx^2 equal to 1. These studies, according to the metric approach, are carried out within the framework of *f*(R) theories of gravity. Our analysis found three solutions based on the assumption of constant scalar curvature. This is how the paper is structured: Section 2 presents the field equations within the framework of the *f*(R) theory of gravity. Section 3 discusses what we discovered regarding five-dimensional plane- symmetric solutions, particularly those with constant curvature. The paper's findings and a summary of the results are presented in the final section.

2. SOME BASICS OF *f*(R) GRAVITY: -

The *f*(R) gravity theory is of significant importance in explaining the evolution of the universe.
It is characterized by an action formulation, which can be expressed as,

$$s = \int \sqrt{-g} \left(\frac{1}{16\pi G} f(R) + L_m \right) d^5 x \tag{1}$$

Here *f*(R) is general function of the Ricci scalar and L_m is the matter of Lagrangian. It should be noted that this action is obtained just by replacing R by *f*(R) in the standard Einstein-Hilbert action.

Upon varying the action (1) with respect to the metric g_{ij}, we obtain the corresponding field equations within the framework of *f*(R) gravity as follows:

$$F(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i\nabla_j F(R) + g_{ij}\Box F(R) = kT_{ij} \tag{2}$$

Where, *F(R)* = $\frac{df(R)}{dR}$, $\Box \equiv \nabla^i\nabla_j$, ∇_i is the covariant derivative, T_{ij} is standard matter energy-momentum tensor derived from the Lagrangian L_m and k (= $\frac{8\pi G}{C^4}$) is the coupling constant in gravitational units.

If we take $f(R) \equiv R$, these equations reduce to the field equations of General Relativity.
On contracting the field equations (2), we get,

$$F(R)R - \frac{5}{2}f(R) + 4\Box F(R) = kT \tag{3}$$

And in vacuum above equation becomes (i.e., energy momentum tensor T_{ij} =0)
From equation (4), we get,

$$F(R)R - \frac{5}{2}f(R) + 4\Box F(R) = 0 \tag{4}$$

From equation (4), we get,

$$f(R) = \frac{2}{5}F(R)R + \frac{8}{5}\Box F(R) \tag{5}$$

This relationship establishes an essential connection between *f*(R) and F(R), and it will play a crucial role in streamlining the field equation and determining the specific form of *f*(R).
From equation (4), it becomes evident that any metric featuring a constant Ricci scalar curvature, denoted as $R = R_0$ can serve as a solution to the contracted equation (4) if the following equation holds:

$$F(R_0)R_0 - \frac{5}{2}f(R_0) = 0 \tag{6}$$

Equation (6) is known as "constant curvature condition".
Further, if we differentiate equation (4) with respect to x gives

$$F'(R)R - \frac{3}{2}R'F(R) + 4(\Box F(R))' = 0 \tag{7}$$

Equation (7) gives consistency relation for *F(R).*

3.FIVE-DIMENSIONAL PLANE SYMMETRIC VACUUM SOLUTIONS: -

In this analysis, we have derived static solutions for a five-dimensional, plane-symmetric spacetime by solving the field equations within the context of *f*(R) gravity. Our approach involves considering both the vacuum field equation and a scenario where the scalar curvature remains constant (R=constant). This yields solutions for a static, higher-dimensional plane-symmetric spacetime.

3.1 Higher dimensional plane symmetric space time:

The line element describing a static, five-dimensional, plane-symmetric spacetime is expressed as follows:

$$ds^2 = A(x)dt^2 - C(x)dx^2 - B(x)(dy^2 + dz^2 + du^2) \tag{8}$$

For simplicity. We take $C(x) = 1$.

$$ds^2 = A(x)dt^2 - dx^2 - B(x)(dy^2 + dz^2 + du^2) \tag{9}$$

Then the corresponding Ricci scalar becomes,

$$R = \frac{3B''}{B} + \frac{A''}{A} - \frac{1}{2}\frac{A'^2}{A^2} + \frac{3}{2}\frac{A'B'}{AB} \tag{10}$$

Where prime denotes the derivative with respect to *x*.
By using equation (4) we have,

$$f(R) = \frac{2}{5}F(R)R + \frac{8}{5}\square F(R) \tag{11}$$

Using this value of *f*(R) in the vacuum field equation, we have,

$$\frac{1}{5}[F(R)R - \square F(R)] = \frac{F(R)R_{ij} - \nabla_i \nabla_j F(R)}{g_{ij}} \tag{12}$$

We have the combination equation is,

$$k_i = \frac{F(R)R_{ij} - \nabla_i \nabla_j F(R)}{g_{ij}} \tag{13}$$

is independent of the index i and hence $k_i - k_j = 0$ for all i and j.
For $k_1 - k_5 = 0$

$$\frac{1}{2}\left[\frac{3}{2}\frac{A'B'}{AB} + \frac{3}{2}\frac{B'^2}{B^2} - 3\frac{B''}{B}\right]F - \frac{1}{2}\left[\frac{A'}{A}F' - 2F''\right] = 0 \tag{14}$$

For $k_2 - k_5 = k_3 - k_5 = k_4 - k_5 = 0$

$$\frac{1}{4}\left[\frac{2A''}{A} - \frac{A'^2}{A^2} + \frac{2A'B'}{AB} - \frac{2B''}{B} - \frac{B'^2}{B^2}\right]F - \frac{1}{2}\left[\frac{A'}{A} - \frac{B'}{B}\right]F' = 0 \tag{15}$$

As a result, we obtain two non-linear equations, equations (14) and (15), which involve three unknowns: A, B, and F. Solving these equations can be a challenging task due to their complexity. Nevertheless, we explore certain solutions by making use of the assumption of constant curvature to simplify the problem.

3.2 Constant curvature solutions:

We assume constant curvature solution, that is for $R=R_0$.
We have,

$$F'(R_0) = 0 = F''(R_0) \tag{16}$$

By using (16), equations (14) and (15) reduced to,

$$\frac{3}{2}\frac{A'B'}{AB} + \frac{3}{2}\frac{B'^2}{B^2} - 3\frac{B''}{B} = 0 \tag{17}$$

$$\frac{2A''}{A} - \frac{A'^2}{A^2} + \frac{2A'B'}{AB} - \frac{2B''}{B} - \frac{B'^2}{B^2} = 0 \tag{18}$$

These equations can be solved by power law assumption, i.e., $A \propto x^m$ and $B \propto x^n$ where m and n are real numbers.
Thus $A = k_1 x^m$ and $B = k_2 x^n$, where k_1 and k_2 are proportionality constants.
From equation (17) and (18),
We get, $m = -1$, $n = 1$. (19)

And hence from (4) and (19) the solution becomes

$$ds^2 = k_1 x^{-1} dt^2 - dx^2 - k_2 x(dy^2 + dz^2 + du^2) \tag{20}$$

It can be shown that these values of m and n gives R=0. We redefine the parameters, i.e., $\sqrt{k_1}t \longrightarrow T$, $\sqrt{k_2}y \longrightarrow Y$, $\sqrt{k_2}z \longrightarrow Z$ and $\sqrt{k_2}u \longrightarrow U$

The above line element takes the form

$$ds^2 = x^{-1} dT^2 - dx^2 - x(dY^2 + dZ^2 + dU^2) \tag{21}$$

Which is exactly same as the well-known Taub's metric [33]
Now assume $A(x) = e^{2\lambda(x)}$ and $B(x) = e^{2\mu(x)}$
So, the line element (9) takes the form

$$ds^2 = e^{2\lambda(x)} dt^2 - dx^2 - e^{2\mu(x)}(dy^2 + dz^2 + du^2) \tag{22}$$

The corresponding Ricci scalar is given by

$$R = 3\mu'' + 6\mu'^2 + 3\mu'\lambda' + \lambda'' + \lambda'^2 \tag{23}$$

Using equation (13), $k_1 - k_5 = 0$ and $k_2 - k_5 = k_3 - k_5 = k_4 - k_5 = 0$ gives

$$3(\mu'\lambda' - \mu'^2 - \mu'')F - \lambda'F' + F'' = 0 \tag{24}$$

$$(-\mu'' - 3\mu'^2 + 2\mu'\lambda' + \lambda'' + \lambda'^2)F + (\mu' - \lambda')F' = 0 \tag{25}$$

For constant curvature solutions, the equations (24) and (25) reduced to

$$\mu'' + \mu'^2 - \mu'\lambda' = 0 \tag{26}$$

$$\mu'' + 3\mu'^2 - 2\mu'\lambda' - \lambda'' - \lambda'^2 = 0 \tag{27}$$

Equation (26) can be written as

$$\mu'\left(\frac{\mu''}{\mu'} + \mu' - \lambda'\right) = 0 \tag{28}$$

The above equations lead to the following two cases:

Case I. $\mu' = 0$,
Case II. $\frac{\mu''}{\mu'} + \mu' - \lambda' = 0$

Case I: -
If $\mu' = 0$
Then $\mu = a$ (29)

Where a is constant of integration. Put this value in equation (27), by integrating that equation we get,

$$\lambda = \ln(xc + bc) \tag{30}$$

Where b and c are constant of integration. Thus, the metric (22) becomes,

$$ds^2 = (xc + bc)^2 dt^2 - dx^2 - e^{2a}(dy^2 + dz^2 + du^2) \tag{31}$$

The corresponding scalar curvature is

$$R=0 \tag{32}$$

It is mentioned here that the metric (31) is a solution only for those *f*(R) functions which are linear superposition of R^m.

For example, $f(R) = R + aR^2$ could be the right choice and
$f(R) = R - (-1)^{n-1}\frac{a}{R^n}$ in this particular case, the assumption of constant curvature, R =0, cannot be applied because the function of the Ricci scalar becomes undefined when R equals zero. It's important to note that this solution corresponds to the self-similar solution of an infinite kind for the parallel dust case [34].

Case II: -
In the case II

$$\lambda' = \mu' + \frac{\mu''}{\mu'} \tag{33}$$

On integrating above equation,

$$\lambda = \mu + \ln \mu' + d \tag{34}$$

Using the assumption of constant scalar curvature, we have,

$$\frac{\mu'''}{\mu'} + 9\mu'' + 10\mu'^2 = constant \tag{35}$$

This is third order non-linear differential equation. The general solution of this equation is very difficult. However, out of larger set of possible solutions choosing $\mu(x)$ is a linear function of x, i.e., $\mu(x) = fx + g$, where f and g are arbitrary constants.
The line element (22) takes the form,

$$ds^2 = e^{2(fx+\bar{g})}dt^2 - dx^2 - e^{2(fx+g)}(dy^2 + dz^2 + du^2) \tag{36}$$

Where $\bar{g} = g + \ln f + d$.

The corresponding Ricci scalar reduced to,

$$R = 20f^2 \tag{37}$$

Now re-defining $e^{\bar{g}}\, t \longrightarrow T, e^{g}\, y \longrightarrow Y, e^{g} z \longrightarrow Z \ and \ e^{g} u \longrightarrow U$

Thus, the metric (22) becomes

$$ds^2 = e^{2fx}(dT^2 - dY^2 - dZ^2 - dU^2) - dx^2 \tag{38}$$

This corresponds to the well-known anti-de Sitter space time [35] in general relativity.

4. SUMMARY AND CONCLUSION: -

In this study, we have explored into higher-dimensional plane-symmetric solutions within the framework of *f*(R) theory of gravity. Our approach involved employing the metric approach of *f*(R) theory to derive solutions for the field equations. Since the field equations are highly non-linear and quite complex, solving them without any assumptions is exceedingly challenging. Consequently, we have introduced a crucial condition, as outlined in equation (6), which focuses on maintaining a constant scalar curvature to facilitate our analysis.

While we have the flexibility to make arbitrary assumptions about the function F in order to address these equations, it's important to note that this approach can lead to highly complex, fifth-order non-linear differential equations. However, the assumption of constant curvature proves to be the most suitable choice in simplifying the problem and allows us to find solutions with a constant scalar curvature.

Through this assumption, we've identified a total of three static plane-symmetric solutions. Among these solutions, one resembles Taub's solution, another corresponds to an anti-de Sitter spacetime, and the third is characterized as a self-similar solution. These solutions hold significant physical relevance in the context of our study.

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