NEW FIXED POINT THEOREMS FOR INTEGRAL TYPE COMPATIBLE MAPPINGS IN CONE METRIC SPACE

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ABSTRACT: *The study of fixed-point theory is very vast area for researchers. It has huge applications in many disciplines of pure and applied part of Mathematics along with Physics, Chemistry, Biology, Computer fields etc. Here we are trying to establish a fixed-point theorem for integral type compatible mapping satisfying integral type contractive inequality in Cone metric space.*

KEYWORDS: Cone metric space, Integral type mapping, Compatible mapping, Coincidence point, Fixed point.

1. INTRODUCTION

Fixed point theory is very extensive and wider field for research. The concept of metric space was introduced by M. Frechet. Banach contraction principle is indeed the most popular result of fixed-point theory, this principle has many applications in several fields like Biology, Physics, Chemistry, Topology, Digital Topology, Fractal theory etc. Branciari gave fixed point result for a single mapping which satisfying Banach contraction condition of integral-type inequality (Branciari, 2002). After that, many researchers have defined some fixed-point theorems involving more general contractive conditions. Huang and Zhang gave the concept of Cone metric space and established fixed point theorems for contractive mappings in cone metric space (Huang & Zhang, 2007). Suzuki shows that Meir-Keer contractions of integral type condition (Suzuki, 2007). Abbas and Jungck generalized the result of two self-maps through weak compatibility in cone metric space (Abbas & Jungck, 2008). Also Rezapour and Hamlbarani proved the normality condition in cone metric space and established fixed point theorem, which is milestone in developing fixed point theory in cone metric space (Rezapour & Hamlbarani, 2008). Wadkar et.al proved some fixed point and common fixed point theorems for integral type mappings in cone metric space which was generalized form of some well known theorems (Wadkar et.al, 2015). In this paper, we establish a fixed-point theorem for weakly compatible mapping satisfying a general contractive inequality of integral type.

2. PRELIMINARIES

In this section we recall some notations and definitions

Definition 2.1(Wadkar et.al, 2015)**:-**let E always be a real Banach space and P is a subset of E. Then P is called a cone in E if,

(i) P is closed, non–empty and $P \neq \{0\}$

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax+by \in P$

(iii) $x \in P$ and $-x \in P$ implies $x=0$ that is $P\cap(-P)=\{0\}$

Given a cone $P\subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x\in P$. We write $x<y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ if and only if $y - x\in IntP$, *Int* denotes the interior of P.

Definition 2.2 (Wadkar et.al, 2015)**:-**The least positive number k satisfying the above condition is called the normal constant of P. The authors showed that there is no normal cone with normal constant $M < 1$ and for each $K > 1$. There is cone with normal constant $M > k$.

Definition2.3 (Wadkar et.al, 2015)**:-**The cone P is called regular if every increasing sequence which is bounded from the above is convergent, that is if$\{x_n\}_{n\geq 1}$ is a sequence such that $x_1\leq x_2 \leq ...\leq y$ for some $y \in X$, then there is $x \in X$ such that $\lim_{n\to\infty} ||x_n-x||= 0$. The cone P is **regular** iff every decreasing sequence which is bounded from below is convergent.

Definition 2.4 (Wadkar et.al, 2015)**:-**Let X a non-empty set and X is a real Banach space, d is a mapping from X into itself such that d satisfies following conditions: -

(d_1) $\int_0^{d(x,y)} \phi(t)dt \geq 0$ for all $x,y\in X$

(d_2) $\int_0^{d(x,y)} \phi(t)dt = 0 \Leftrightarrow x = y$

(d_3) $\int_0^{d(x,y)} \phi(t)dt = \int_0^{d(y,x)} \phi(t)dt$

(d_4) $\int_0^{d(x,z)} \phi(t)dt \leq \int_0^{d(x,y)} \phi(t)dt + \int_0^{d(y,z)} \phi(t)dt$

Then d is called a cone metric on X and(X, d) is called Cone Metric Space

Definition 2.5 (Wadkar et.al, 2015)**:-** Let *A & S* be a two mappings of a cone metric space *(X, d)* then it is said to be compatible if $\lim_{n\to\infty}\int_0^{d(ASx_n,\, SAx_n)}\phi(t)dt$ =0 whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} Ax_n = t$ and $\lim_{n\to\infty} Sx_n = t$ for some $t \in X$ of cone metric space

Definition 2.6 (Wadkar et.al, 2015)**:-** Let *A* and *S* are two self-mappings of *(X, d*) then these are said to be weakly compatible, if they commute at a coincidence point, that is *Sx=Ax* implies that

$SAx = ASx \qquad \forall\, x \in X$..

It is easy to see that compatible mapping commutes at their coincidence points. It is note that compatible maps are weakly compatible but converse need not be true for any $x \in X$.

3. MAIN RESULT

Here we will establish compatibility and contractive conditions for self-mappings *A, B, C* & *D* in complete cone metric space without assuming their normality.

Theorem 3.1:- Let *(X,d)* be a complete cone metric space with normal cone *P*. Let *A*,*B*, *C* and *D* be four self-mappings from *X* to *X* satisfying the condition

$$\int_0^{d(Ax,By)}\emptyset(t)dt \quad \le \lambda\int_0^{d(Ax,Cx)}\emptyset(t)dt + \mu\int_0^{d(By,Dy)}\emptyset(t)dt + \delta\int_0^{d(Cx,Dy)}\emptyset(t)dt$$

$$+\gamma\int_0^{d(Ax,Dy)+d(Cx,By)}\emptyset(t)dt$$

For all $x, y \in X;\ \lambda, \mu, \delta, \eta \in [0,1/2)$,with $\lambda+\mu+\delta+2\gamma < 1$

(i).If *A(X)* ⊆*D(X)* & *B(X)* ⊆*C(X)*

(ii).*(A,C)* and *(B,D)* are weakly compatible mapping.

(iii).*A* and *C* are continuous then *A,B,C* and *D* have unique common fixed point in *X*.

(iv)-For some $\lambda, \mu, \delta, \eta \in [0,1/2)$ with $\lambda+\mu+\delta+2\gamma < 1$ Such that $x, y \in X$.

Then *A*,*B*,*C* and *D* have a unique common fixed point in *X*.

Proof:-Let $x_0 \in X$ be any point, now construct a sequence $\{x_k\}$ and $\{y_k\}$ such that

$Ax_k = Dx_{2k+1} = y_{2k}$, $Bx_{2k+1} = Cx_{2k+2} = y_{2k+1}$; $k \in N$..........*(3.1.1)*

Now we consider,

$$\int_0^{d(y_{2k},y_{2k+1})} \phi(t)dt = \int_0^{d(Ax_{2k},Bx_{2k+1})} \phi(t)dt$$

$$\leq \lambda \int_0^{d(Ax_{2k},Bx_{2k+1})} \phi(t)dt + \mu \int_0^{d(Bx_{2k+1},Dx_{2k+1})} \phi(t)dt$$

$$+\delta \int_0^{d(Cx_{2k},Dx_{2k+1})} \phi(t)\mathrm{d}t + \lambda \int_0^{d(Ax_{2k},Dx_{2k+1})+d(Cx_{2k},Bx_{2k+1})} \phi(t)dt$$

$$\leq \lambda \int_0^{d(y_{2k},y_{2k+1})} \phi(t)\mathrm{d}t + \mu \int_0^{d(y_{2k+1},y_{2k})} \phi(t)\mathrm{d}t$$

$$+\delta \int_0^{d(y_{2k-1},y_{2k})} \phi(t)\mathrm{d}t + \gamma \int_0^{d(y_{2k},y_{2k})+d(y_{2k-1},y_{2k+1})} \phi(t)\mathrm{d}t$$

$$\leq \lambda \int_0^{d(y_{2k},y_{2k+1})} \phi(t)\mathrm{d}t + \mu \int_0^{d(y_{2k+1},y_{2k})} \phi(t)\mathrm{d}t$$

$$+\delta \int_0^{d(y_{2k-1},y_{2k})} \phi(t)\mathrm{d}t + \gamma \int_0^{d(y_{2k-1},y_{2k})+\mathrm{d}(y_{2k},y_{2k+1})} \phi(t)\mathrm{d}t$$

$$\Rightarrow (1-\mu-\gamma) \int_0^{d(y_{2k},y_{2k+1})} \phi(t)\mathrm{d}t \leq \lambda \int_0^{d(y_{2k},y_{2k-1})} \phi(t)\mathrm{d}t + \delta \int_0^{d(y_{2k-1},y_{2k})} \phi(t)\mathrm{d}t$$

$$+ \gamma \int_0^{d(y_{2k-1},y_{2k})} \phi(t)\mathrm{d}t$$

$$\Rightarrow \int_0^{d(y_{2k},y_{2k+1})} \phi(t)\mathrm{d}t \leq \frac{\lambda+\delta+\gamma}{1-\mu-\gamma} \int_0^{d(y_{2k-1},y_{2k})} \phi(t)\mathrm{d}t$$

$$\Rightarrow \int_0^{d(y_{2k},y_{2k+1})} \phi(t)\,\mathrm{d}t \le h \int_0^{d(y_{2k-1},y_{2k})} \phi(t)\,\mathrm{d}t \qquad \text{....................(3.1.2)}$$

Where $h = \frac{\lambda+\delta+\gamma}{1-\mu-\gamma} < 1$

Now we let,

$$Ax_{2k+2} = Dx_{2k+3} = y_{2k+2},\ Bx_{2k+3} = Cx_{2k+4} = y_{2k+3}$$

$$\int_0^{d(Ax_{2k+2},Bx_{2k+2})} \phi(t)\,\mathrm{d}t \le \lambda \int_0^{d(Ax_{2k+2},Cx_{2k+2})} \phi(t)\,\mathrm{d}t + \mu \int_0^{d(Bx_{2k+1},Dx_{2k+1})} \phi(t)\,\mathrm{d}t$$

$$+\delta \int_0^{d(Cx_{2k+2},Dx_{2k+1})} \phi(t)\,\mathrm{d}t$$

$$+\gamma \int_0^{d(Ax_{2k+2},Dx_{2k+1})+d(Cx_{2k+2},Bx_{2k+1})} \phi(t)\,\mathrm{d}t$$

By using equation (3.1.1) we get-

$$\int_0^{d(y_{2k+2},y_{2k+1})} \phi(t)\,\mathrm{d}t = \int_0^{d(Ax_{2k+2},Bx_{2k+1})} \phi(t)\,\mathrm{d}t$$

$$\le \lambda \int_0^{d(y_{2k+2},y_{2k+1})} \phi(t)\,\mathrm{d}t + \mu \int_0^{d(y_{2k+1},y_{2k})} \phi(t)\,\mathrm{d}t$$

$$+\delta \int_0^{d(y_{2k+1},y_{2k})} \phi(t)\,\mathrm{d}t + \gamma \int_0^{d(y_{2k+2},y_{2k})+d(y_{2k+1},y_{2k+1})} \phi(t)\,\mathrm{d}t$$

$$\le \lambda \int_0^{d(y_{2k+2},y_{2k+1})} \phi(t)\,\mathrm{d}t + \mu \int_0^{d(y_{2k+1},y_{2k})} \phi(t)\,\mathrm{d}t$$

$$+ \delta \int_0^{d(y_{2k+1},y_{2k})} \phi(t)dt + \gamma \int_0^{d(y_{2k+2},y_{2k+1})+d(y_{2k+1},y_{2k})} \phi(t)\,\mathrm{d}t$$

$$\Rightarrow (1-\lambda-\gamma) \int_0^{d(y_{2k+2},y_{2k+1})} \phi(t)\,\mathrm{d}t \le \mu \int_0^{d(y_{2k+1},y_{2k})} \phi(t)\,\mathrm{d}t + \delta \int_0^{d(y_{2k+1},y_{2k})} \phi(t)\,\mathrm{d}t$$

$+ \gamma \int_0^{d(y_{2k+1}, y_{2k})} \phi(t)\,\mathrm{d}t$

$\Rightarrow \int_0^{d(y_{2k+2}, y_{2k+1})} \phi(t)\,\mathrm{d}t \le \frac{\mu+\delta+\gamma}{1-\lambda-\gamma} \int_0^{d(y_{2k+1}, y_{2k})} \phi(t)\,\mathrm{d}t$

$$h = \frac{\mu+\delta+\gamma}{1-\lambda-\gamma} < 1$$

$(1-\lambda-\gamma) \int_0^{d(y_{2k+2}, y_{2k+1})} \phi(t)\,\mathrm{d}t \;\le h \int_0^{d(y_{2k+1}, y_{2k})} \phi(t)\,\mathrm{d}t$ where $h = \frac{\mu+\delta+\gamma}{1-\lambda-\gamma} < 1$

$$\le h.h \int_0^{d(y_{2k-1}, y_{2k})} \phi(t)\,\mathrm{d}t$$

$\le .... \le .... \le ...$

$$\le h^{2k+1} \int_0^{d(y_0, y_1)} \phi(t)\,\mathrm{d}t$$

Where$h = \frac{\mu+\delta+\gamma}{1-\lambda-\gamma} < 1$,

Now for any $m > k$

$\int_0^{d(y_{2k}, y_{2m})} \phi(t)\,\mathrm{d}t \le \int_0^{d(y_{2k}, y_{2k+1}) + d(y_{2k+1}, y_{2k}) + --- + d(y_{2m-1}, y_{2m})} \phi(t)\,\mathrm{d}t$

$$\le \int_0^{d(y_{2k}, y_{2k-1})} \phi(t)\,\mathrm{d}t + \int_0^{d(y_{2k+1}, y_{2k+2})} \phi(t)\,\mathrm{d}t + ...$$

$.......... + \int_0^{d(y_{2m-1}, y_{2m})} \phi(t)\,\mathrm{d}t$

$$\le h^{2k} \int_0^{d(y_0, y_1)} \phi(t)\,\mathrm{d}t + h^{2k+1} \int_0^{d(y_0, y_1)} \phi(t)\,\mathrm{d}t + \ldots \ldots \ldots$$

$$\le (h^{2k} + h^{2k+1} + h^{2k+2} + ...) \int_0^{d(y_0, y_1)} \phi(t)\,\mathrm{d}t$$

$$\leq \frac{h^{2k}}{1-h}\int_0^{d(y_0,y_1)} \phi(t)\,\mathrm{d}t$$

$$\Rightarrow \| \int_0^{d(y_{2n},y_{2m})} \phi(t)dt \| \leq \frac{h^{2k}}{1-h}\int_0^{d(y_0,y_1)} \phi(t)\,\mathrm{d}t$$

$$\Rightarrow \lim_{n\to\infty} \| \int_0^{d(y_{2n},y_{2m})} \phi(t)dt \| \to 0$$

$$\Rightarrow \lim_{n\to\infty} \| \int_0^{d(y_{2k+2},y_{2k+1})} \phi(t)dt \| \to 0 \quad \text{as } n\to\infty$$

Since $\{x_k\}$ is a Cauchy sequence, therefore $X$is complete which converges to $u \in X$.

Hence *(X,d)* is complete cone metric space.

Thus, $x_k \to u$ as $n \to \infty$ and $\{Ax_{2k}\} \to u$ and $\{Bx_{2k+1}\} \to u$.

Case 1:- Map C is continuous

As C is continuous we have:-

$C^2x_{2k} \to Cu$, $CAx_{2k} \to Cu$

Since (A,C) is compatible, so we have $ACx_{2k} \to Cu$

Now,

$$\int_0^{d(Cu,u)} \phi(t)dt \leq \int_0^{d(Cu,ACx_{2k})} \phi(t)dt + \int_0^{d(ACx_{2k},Bx_{2k+1})} \phi(t)dt$$

$$+ \int_0^{d(Bx_{2k+1},u)} \phi(t)dt$$

$$= \int_0^{d(Su,ACx_{2k})} \phi(t)dt + \int_0^{d(y_{2k+1},u)} \phi(t)dt$$

$$+\int_0^{d(ACx_{2k},Bx_{2k+1})}\phi(t)dt$$

Where $x = Cx_{2k}$ & $y = x_{2k+1}$

$$\int_0^{d(Cu,u)}\phi(t)dt \le \int_0^{d(Cu,ACx_{2k})}\phi(t)dt + \int_0^{d(y_{2k+1},u)}\phi(t)dt$$

$$+\lambda\int_0^{d(ACx_{2k},C^2x_{2k})}\phi(t)dt \ +\mu\int_0^{d(Bx_{2k+1},Dx_{2k+1})}\phi(t)dt$$

$$+\delta\int_0^{d(C^2x_{2k},Dx_{2k+1})}\phi(t)dt$$

$$+\gamma\int_0^{d(ACx_{2k},Dx_{2k+1})+d(Bx_{2k+1},C^2x_{2k})}\phi(t)dt$$

$$=\int_0^{d(Cu,ACx_{2k})}\phi(t)dt + \int_0^{d(y_{2k+1},u)}\phi(t)dt$$

$$+\lambda\int_0^{d(ACx_{2k},C^2x_{2k})}\phi(t)dt + \mu\int_0^{d(y_{2k+1},y_{2k})}\phi(t)dt$$

$$+\delta\int_0^{d(C^2x_{2k},y_{2k})}\phi(t)dt + \gamma\int_0^{d(ACx_{2k},Cu)+d(Cu,C^2x_{2k})}\phi(t)dt$$

$$\le \int_0^{d(Cu,ACx_{2k})}\phi(t)dt + \int_0^{d(y_{2k+1},u)}\phi(t)dt$$

$$+\lambda\int_0^{d(ACx_{2k},Cu)+d(Cu,C^2x_{2k})}\phi(t)dt + \int_0^{d(y_{2k+1},u)+d(u,y_{2k})}\phi(t)dt$$

$$+\delta\int_0^{d(C^2x_{2k},Cu)+d(Cu,u)+d(u,y_{2k})}\phi(t)dt$$

$$+\gamma\int_0^{d(ACx_{2k},Cu)+d(Cu,u)+d(u,y_{2k})+d(y_{2k+1},u)+d(Cu,u)+d(Cu,C^2x_{2k})}\phi(t)dt$$

$$\Rightarrow [1-\delta-2\gamma]\int_0^{d(Cu,u)}\phi(t)dt \le [1+\lambda+\gamma]\int_0^{d(Cu,ACx_{2k})}\phi(t)dt$$

$$+[\lambda+\delta+\gamma]\int_0^{d(Cu,C^2x_{2k})}\phi(t)dt$$

$$+[1+\mu+\gamma]\int_0^{d(y_{2k+1},u)}\phi(t)dt+[\mu+\delta+\gamma]\int_0^{d(u,y_{2k})}\phi(t)dt$$

$ACx_{2k} \to Cu,\ C^2x_{2k} \to Cu$

$\{y_{2k}\} \to u \quad \& \{y_{2k+1}\} \to u$

So we have, $d(Cu, u) = 0$, we get $Cu = u$

Now,

$$\int_0^{d(Au,Cu)}\phi(t)dt \le \int_0^{d(Au,Bx_{2k+1})}\phi(t)dt+\int_0^{d(Bx_{2k+1},Cu)}\phi(t)dt$$

$$=\int_0^{d(Cu,ACx_{2k+1})}\phi(t)dt+\int_0^{d(y_{2k+1},Cu)}\phi(t)dt$$

$$+\int_0^{d(Cu,ACx_{2k+1})}\phi(t)dt$$

Let $x = u$, & $y = x_{2k+1}$, we have

$$\int_0^{d(Au,Cu)}\phi(t)dt \le \int_0^{d(y_{2k+1},Cu)}\phi(t)dt+\lambda\int_0^{d(Au,Cu)}\phi(t)dt$$

$$+\mu\int_0^{d(Bx_{2k+1},Dx_{2k+1})}\phi(t)dt$$

$$+\delta\int_0^{d(Cu,Dx_{2k+1})}\phi(t)dt+\gamma\int_0^{d(Au,Dx_{2k+1})+d(Bx_{2k+1},Cu)}\phi(t)dt$$

$$=\int_0^{d(y_{2k+1},Cu)}\phi(t)dt+\lambda\int_0^{d(Au,Dx_{2k+1})+d(Bx_{2k+1},Cu)}\phi(t)dt$$

$$+\mu\int_0^{d(y_{2k+1},y_{2k})}\phi(t)dt\ +\delta\int_0^{d(Cu,y_{2k})}\phi(t)dt$$

$$+\gamma\int_0^{d(Au,y_{2k})+d(y_{2k+1},y_{2k})}\phi(t)dt$$

$$\leq\int_0^{d(y_{2k+1},Cu)}\phi(t)dt+\lambda\int_0^{d(Au,Cu)}\phi(t)dt$$

$$+\mu\int_0^{d(y_{2k+1},Cu)+d(Cu,y_k)}\phi(t)dt+\delta\int_0^{d(Cu,y_{2k})}\phi(t)dt$$

$$+\gamma\int_0^{d(Au,Cu)+d(Cu,y_{2k})+d(y_{2k+1},Cu)}\phi(t)dt$$

$$\Rightarrow[1\text{-}\lambda\text{-}\gamma]\int_0^{d(Au,Cu)}\phi(t)dt\leq(u+\delta+\gamma)\int_0^{d(y_{2k},Cu)}\phi(t)dt+(1+\mu)+\int_0^{d(y_{2k+1},Cu)}\phi(t)dt$$

Using $Cu = u$ than we have,

$$[1\text{-}\lambda\text{-}\gamma]\int_0^{d(Au,Cu)}\phi(t)dt\leq(u+\delta+\gamma)\int_0^{d(y_{2k},u)}\phi(t)dt+(1+\mu+\gamma)\int_0^{d(y_{2k+1},u)}\phi(t)dt$$

As $\{y_{2k}\}\to u\ \&\ \{y_{2k+1}\}\to u$

We have $d(Au,u)=0$

And, $Au = Cu = u$.

Thus u is a point of coincidence of the pair of maps (A,C).

As $A(x)\subseteq D(x)$ there exists $v\in X$ such that $u=Au=Dv$ so, $u=Au=Cu=Dv$

Taking $x = u$, $y = v$,

$$\int_0^{d(Au,Bv)} \phi(t)dt \le \lambda \int_0^{d(Au,Cu)} \phi(t)dt + \mu \int_0^{d(Bv,Dv)} \phi(t)dt$$

$$+\delta \int_0^{d(Cu,Dv)} \phi(t)dt + \gamma \int_0^{d(Au,Dv)+d(Bv,Cu)} \phi(t)dt$$

So, we have,

$$\int_0^{d(u,Bv)} \phi(t)dt \le (\mu+\gamma) \int_0^{d(u,Bv)} \phi(t)dt$$

As $\mu+\gamma<1$ it follows that $d(Bv,u)=0$

And we get $Bv=u$

Thus $Bv=Dv=u$ as their (B,D) is weak compatible we get $Bu=Du$

Taking $x=u$, $y=u$ using $Au=Cu$ &$Bu=Du$ we get

$$\int_0^{d(Au,Bu)} \phi(t)dt \le (\delta+2\gamma) \int_0^{d(Au,Bu)} \phi(t)dt$$

$Au=Bu$

As $\delta+2\gamma<1$ and we have $u=Au=Cu=Bu=Du$

Thus u is a point of coincidence of the four self-maps$A,B,C,$and D in this case.

Case 2:- Map A is continuous

As C is continuous, we have

$A^2x_{2k}\to Au$, $ACx_{2k}\to Au$

Since (A,C) is compatible, so we have $CAx_{2k}\to Au$

Now,

$$\int_0^{d(Au,u)} \phi(t)dt \le \int_0^{d(Au,A^2x_{2k})} \phi(t)dt + \int_0^{d(A^2x_{2k},Bx_{2k+1})} \phi(t)dt + \int_0^{d(B_{2k+1},u)} \phi(t)dt$$

Similarly, we can prove

$$[1-\delta-2\gamma]\int_0^{d(Au,u)}\phi(t)dt \le (1+\lambda+\gamma)\int_0^{d(Au,A^2x_{2k})}\phi(t)dt$$

$$+ (\lambda+\delta+\gamma)\int_0^{d(Au,CAx_{2k})}\phi(t)dt + (\mu+\delta+\gamma)\int_0^{d(y_{2k},u)}\phi(t)dt$$

$$+(1+\mu+\gamma)\int_0^{d(y_{2k},u)}\phi(t)dt$$

As $CAx_{2A} \to Au$, $A^2x_{2k} \to Au$

$\{y_{2k}\} \to u$ & $\{y_{2k+1}\} \to u$

So we have, $d(Au, u) = 0$, we get $Au = u$

$A(x) \subseteq D(x)$ there exist $v_1 \in X$ such that, $U=Au=Dv_1$

$$\int_0^{d(u,Bv_1)}\phi(t)dt \le \int_0^{d(Ax_{2k},Bv_1)}\phi(t)dt + \int_0^{d(Ax_{2k},u)}\phi(t)dt$$

Taking $x = x_{2k}$, $y = v_1$ then $u = Tv_1$

$$[1-\mu-\gamma]\int_0^{d(Bv_1,u)}\phi(t)dt \le (1+\lambda+\gamma)\int_0^{d(u,y_{2k})}\phi(t)dt + (\gamma+\lambda+\delta)\int_0^{d(u,y_{2k-1})}\phi(t)dt$$

$$\int_0^{d(u,Bu)}\phi(t)dt \le \int_0^{d(y_{2k},u)}\phi(t)dt + \lambda\int_0^{d(Ax_{2k},Cx_{2k})}\phi(t)dt$$

$$+\mu\int_0^{d(Au,Du)}\phi(t)dt$$

$$+\delta\int_0^{d(C_{2k+1},Du)}\phi(t)dt + \gamma\int_0^{d(Au,Du)+d(Cx_{2k},Bu)}\phi(t)dt$$

$$[1-\delta-2\gamma]\int_0^{d(Bu,u)}\phi(t)dt \le (1+\lambda+\gamma)\int_0^{d(u,y_{2k})}\phi(t)dt + (1+\gamma+\delta)\int_0^{d(u,y_{2k-1})}\phi(t)dt$$

As $\{y_{2k}\} \to u \, \& \{y_{2k+1}\} \to u$

We have *d(u,Bu)=0*

And we get *Bu* = u thus *u=Bu=Du=Au*

Now as *B(x)⊆C(x)* there exists $w_1 \epsilon X$ such that,$U=Bu=Cw_1$

$$\int_0^{d(Aw_1,u)} \phi(t)dt = \int_0^{d(Aw_1,Bu)} \phi(t)dt$$

$x = w_1$, $y = u$ with $u=Du=Bu=Cw_1$

$$\int_0^{d(Aw_1,Bu)} \phi(t)dt \le \lambda \int_0^{d(Aw_1,cw_1)} \phi(t)dt + \mu \int_0^{d(Bu,Du)} \phi(t)dt$$

$$+\delta \int_0^{d(cw_1,Du)} \phi(t)dt + \gamma \int_0^{d(Aw_1,Du)+d(Bu,Cw_1)} \phi(t)dt$$

$$= \lambda \int_0^{d(Aw_1,u)} \phi(t)dt + \mu \int_0^{d(u,u)} \phi(t)dt$$

$$+\delta \int_0^{d(u,u)} \phi(t)dt + \gamma \int_0^{d(Aw_1,u)+d(u,u)} \phi(t)dt$$

$$= \lambda \int_0^{d(Aw_1,u)} \phi(t)dt + \gamma \int_0^{d(Aw_1,u)} \phi(t)dt$$

So, $\int_0^{d(Aw_1,u)} \phi(t)dt \le (\lambda+\gamma) \int_0^{d(Aw_1,u)} \phi(t)dt$

Hence $AW_1 = u$ as, $\lambda+\gamma < 1$

$Aw_1 = Cw_1 = u$ as *(A,C)* is compatible so *(A,C)* is weakly compatible.

$Au = Cu$ thus $u = Au = Bu = Cu = Du$

Hence *u* is common fixed point of the four self-maps in both the cases.

UNIQUENESS

Let w = $Bw = Cw = Dw$be another common fixed point of four self-maps.

Taking $x = u$ and $y = w$ than we get

$$\int_0^{d(Au,Bw)} \phi(t)dt \le \lambda \int_0^{d(Au,Cu)} \phi(t)dt + \mu \int_0^{d(Bw,Dw)} \phi(t)dt$$

$$+\delta \int_0^{d(Cu,Du)} \phi(t)dt + \gamma \int_0^{d(Au,Dw)+d(Bw,Cu)} \phi(t)dt$$

$$\Rightarrow \int_0^{d(u,w)} \phi(t)dt \le [\delta+2\gamma] \int_0^{d(u,w)} \phi(t)dt$$

Hence $u = w$ as $\delta+2\gamma < 1$.

Thus, the four self-mapsA,B,C & D have a unique common fixed point.

4.CONCLUSION:-We introduced the concept of compatibility of pair of self-maps in cone metric space without assuming its normality. By Using this concept, we establish a unique common fixed-point theorem for integral type compatible mapping in which four self-mappings satisfying a general contractive condition in cone metric space.

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