GENERAL STUDY ON POLYNOMIALS ASSOCIATED WITH SINHA POLYNOMIALS

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ABSTRACT

The main object of this present paper is to provide a unified presentation of class of Sinha polynomials which generalizes the well known class of Gegenbauer, Legendre polynomials. In this paper we shall give some basic relations involving the generalized Sinha polynomials and then take up several operational result then we obtain series representation, hypergeometric representation and generating function which are best stated in terms of the generalized polynomial.

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INTRODUCTION

Gould [7] presented a systematic study of an interesting generalization of Sinha, Gegenbauer and several other polynomials systems defined by

$$\sum_{n=0}^{\infty} P_n(m, x, y, p, c)t^n = (c - mxt + yt^m)^p. \qquad \ldots(1.1)$$

where m is a positive integer and other parameters are unrestricted in general. In [4], [5] Milovanovic and Dordevic considered the polynomial $\{P_{n,m}^{\lambda}\}_{n=0}^{\infty}$ defined by the generating function,

$$G_m^{\lambda}(x, t) = (1 + 2xt + t^m)^{-\lambda}$$

$$= \sum_{n=0}^{\infty} P_{n,m}^{\lambda}(x)t^n, \qquad \ldots(1.2)$$

where $m \in N$ and $\lambda > -1/2$.

Note that

$$\text{For } m=1, P^{\lambda}_{n,1}(x)=\frac{(\lambda)_n}{n!}(2x-1)^n,$$

which is the Horadam polynomials [1].

For m = 2, $P^{\lambda}_{n,2}(x)=C^{\lambda}_{n}(x)$ (Gogenbauer Polynomials)

For m = 3, weget Horadam-Pethe Polynomials [2]

i.e. $p^{\lambda}_{n,3}(x)=P^{\lambda}_{n+1}(x),$

where $(\lambda)_0=1,$

$$(\lambda)_n=(\lambda+1)(\lambda+2)(\lambda+3)\ldots(\lambda+n-1) \quad , \qquad n=1,2,3,\ldots$$

The polynomial $P^{\lambda}_{n,m}(x)$ is defined by

$$P^{\lambda}_{n,m}(x)=\sum_{k=0}^{[n/m]}\frac{(-1)^k(\lambda)_{n-(m-1)k}(2x)^{n-mk}}{k!(n-mk)!} \qquad \ldots(1.3)$$

The set of polynomials denoted by $S^{\nu}_{n}(x)$ considered by Sinha [13]

$$\sum_{n=0}^{\infty} S^{\nu}_{n}(x)\, t^n=[1-2xt+t^2(2x-1)]^{-\nu}, \qquad \ldots(1.4)$$

where $S^{\nu}_{n}(x)$ is defined by

$$S^{\nu}_{n}(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}(2x-\alpha)^k}{k!(n-2k)!} \qquad \ldots(1.5)$$

$$S^{\nu}_{n}(x)=\sum_{k=0}^{[n/2]}\frac{(2\nu)_n(x-1)^{2k}x^{n-2k}}{2^{2k}\left(1+\frac{1}{2}\right)_k(n-2k)!\,k!} \qquad \ldots(1.6)$$

Polynomials (1.4) is precisely a generalization of $S_n(x)$ defined and studied by Shreshtha [9].

For $\nu = 1/2$, then (1.4) gives associated Legendre polynomials.

A generalization and unification of various polynomials mentioned above is provided by the definition

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x, y)\, t^n = [c - 2xt + t^2 (2x - y)^a]^{-\nu} \qquad \ldots(1.7)$$

For c = y = a = 1, then (1.7) reduces to (1.4)

i.e. $$\phi_n^{\nu}(x,1) = S_n^{\nu}(x) \qquad \ldots(1.8)$$

For c = 1, a = 0, $\nu = \lambda$, then by (1.7), we get

$$\phi_n^{\nu}(x, y) = P_{n,2}^{\lambda}(x) = C_n^{\lambda}(x) \quad \text{(Gegenbauer polynomials)} \qquad \ldots(1.9)$$

Now by Pathan and Khan [8], Srivastava and Manocha [6]

$$(n - mk)! = \frac{(-1)^{mk}}{(-n)_{mk}}\, n!\, ;\ 0 \le mk \le n \qquad \ldots(1.10)$$

Now by [8; p.55 (2.3 & 2.5), p.57 (3.2); p.58 (3.3)]

$$(\alpha)_{n\text{-}k} = \frac{(-1)^k (\alpha)_n}{(1 - \alpha - n)_k};\ 0 \le k \le n \qquad \ldots(1.11)$$

$$(1 - z)^{-a} = {}_1F_0(a; -; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{n!} \qquad \ldots(1.12)$$

and

$$(t + \nu)^n = \sum_{k=0}^{\infty} \frac{n!\, t^k \nu^{n-k}}{k!(n - k)!} \qquad \ldots(1.13)$$

We know that Legendre's duplication formula is

$$(n)_{2k} = 2^{2k} \left(\frac{n}{2}\right)_k \left(\frac{n+1}{2}\right)_k;\ k = 0,1,2,\ldots \qquad \ldots(1.14)$$

In the present paper, we shall give some basic relation involving the generalized polynomial $\phi_n^{\nu}(x, y)$ and then take up several operational results, series representations, hypergeometric representations and generating functions.

Definition (1.7) of $\phi_n^{\nu}(x, y)$ is general enough to account for many of polynomials involved in generalized potential problems [10], [11], [12].

This is interesting since as will be shown the polynomial $\phi_n^{\nu}(x, y)$ contain [7], [5], [13].

Finite series representation for ϕ_n^{ν}(x,y)

Here we obtain the following two finite series representation for $\phi_n^{\nu}(x, y)$, viz

(i) $$\phi_n^{\nu}(x, y)=\sum_{k=0}^{[n/2]} \frac{(-1)^k c^{-\nu-n+k}(\nu)_{n-k}(2x)^{n-2k}(2x-y)^{ak}}{k!(n-2k)!} \quad \ldots(2.1)$$

(ii) $$\phi_n^{\nu}(x, y)=\sum_{k=0}^{[n/2]} \sum_{s=0}^{k} \frac{c^{-\nu-n+s}(\nu)_k (2k+2\nu)_{n-2k}(-k)_s}{k!\, s!\, (n-2k)!} x^n \left(\frac{(2x-y)^a}{x^2}\right)^s \quad \ldots(2.2)$$

Proof.(i) Now by (1.7)

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x, y)t^n = [c-2xt+t^2(2x-y)^a]^{-\nu}$$

$$= c^{-\nu}\left[1-\left\{\frac{2xt-t^2(2x-y)^a}{c}\right\}\right]^{-\nu}$$

Now using (1.12) and (1.13), we get

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x, y)t^n = c^{-\nu}\sum_{n=0}^{\infty} (\nu)_n \sum_{k=0}^{\infty} \frac{(-1)^k (2xt)^{n-k}[(2x-y)^a t^2]^k}{k!(n-k)!c^n}$$

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x, y)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(-1)^k c^{-\nu-n+k}(\nu)_{n-k}(2x)^{n-2k}(2x-y)^{ak} t^n}{k!(n-2k)!}$$

Now equating the coefficient of t^n on both side, we arrive at (2.1).

Proof (ii)

Again by (1.7)

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x, y)t^n = [c - 2xt + t^2(2x - y)^a]^{-\nu}$$

$$= c^{-\nu}\left[1 - \frac{2xt}{c} + \left(\frac{xt}{c}\right)^2 - \left(\frac{xt}{c}\right)^2 + \frac{t^2(2x-y)^a}{c}\right]^{-\nu}$$

$$= c^{-\nu}\left(1 - \frac{xt}{c}\right)^{-2\nu}\left[1 - \left\{\frac{\left(\frac{xt}{c}\right)^2 - \frac{t^2}{c}(2x-y)^a}{\left(1-\frac{xt}{c}\right)^2}\right\}\right]^{-\nu}$$

Now using (1.12), we get

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x, y)t^n$$

$$= c^{-\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{s=0}^{k} \frac{(\nu)_k (2k+2\nu)_n (-k)_s}{k!\, n!\, s!}\left(\frac{xt}{c}\right)^{n+2k}\left(\frac{c(2x-y)^a}{x^2}\right)^s$$

$$\sum \phi_n^{\nu}(x, y)t^n$$

$$= c^{-\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{[n/2]}\sum_{s=0}^{k} \frac{(\nu)_k (2k+2\nu)_{n-2k} (-k)_s}{k!\ \ s!(n-2k)!}\left(\frac{xt}{c}\right)^{n}\left(\frac{c(2x-y)^n}{x^2}\right)^s$$

Comparing the coefficient of t^n on both side, we arrive at (2.2).

Hypergeometric Representation for ϕ_n^{ν} (x,y)

The finite series representation (2.1) for $\phi_n^{\nu}(x, y)$ is of particular interest to us in obtaining the following hypergeometric form for $\phi_n^{\nu}(x, y)$, viz.

$$\phi_n^{\nu}(x,y)=\frac{(\nu)_n(2x)^n}{n!c^{n+\nu}}\ {}_2F_1\left[\begin{matrix}\frac{-n}{2},\frac{-n+1}{2};\\ 1-\nu-n;\end{matrix}\ \frac{c(2x-y)^a}{x^2}\right] \qquad \ldots(3.1)$$

Proof

Since by (2.1)

$$\phi_n^{\nu}(x,y)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(\nu)_{n-k}(2x)^{n-2k}(2x-y)^{ak}}{k!(n-2k)!c^{\nu+n-k}}$$

Now using (1.10) and (1.11), we get

$$\phi_n^{\nu}(x,y)=\sum_{k=0}^{[n/2]}\frac{(\nu)_n(-n)_{2k}}{k!(1-\nu-n)_k\,n!}\frac{(2x)^{n-2k}(2x-y)^{ak}}{c^{\nu+n-k}}$$

Now using Legendre's duplication formula, we get

$$\phi_n^{\nu}(x,y)=\sum_{k=0}^{[n/2]}\frac{(\nu)_n\,2^{2k}\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k}{k!(1-\nu-n)_k\,n!}\frac{(2x)^{n-2k}(2x-y)^{ak}}{c^{\nu+n-k}}$$

$$=\frac{(\nu)_n(2x)^n}{n!c^{n+\nu}}\sum_{k=0}^{[n/2]}\frac{(\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k}{(1-\nu-n)_k\,k!}\left(\frac{c(2x-y)^a}{x^2}\right)^k$$

which is equivalent to (3.1).

Generating Functions for ϕ_n^{ν} (x,y)

We now obtain the following generating function for $\phi_n^{\nu}(x,y)$, viz

$$\sum_{n=0}^{\infty}\frac{(e)_n\phi_n^{\nu}(x,y)t^n}{(\nu)_n}$$

$$=\sum_{n=0}^{\infty}\frac{(e)_n(2xt)^n}{n!c^{\nu+n}}\ {}_3F_2\left[\begin{matrix}\frac{e+n}{2},\frac{e+n+1}{2},\nu+n;\\ \frac{\nu+n}{2},\frac{\nu+n+1}{2};\end{matrix}\ \frac{-(2x-y)^a t^2}{c}\right] \qquad \ldots(4.1)$$

Proof. Consider the following series and using (2.1), we get

$$\sum_{n=0}^{\infty} \frac{(e)_n \phi_n^{\nu}(x,y)t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(e)_n (-1)^k (\nu)_{n-k} (2x)^{n-2k}}{(\nu)_n \, k!(n-2k)!} \frac{(2x-y)^{ak} t^n}{c^{\nu+n-k}}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(e)_{n+2k} (-1)^k (\nu)_{n+k} (2x)^n (2x-y)^{ak} t^{n+2k}}{(\nu)_{n+2k} \, k!\, n!\, c^{\nu+n+k}}$$

Now using (1.8) and Legendre's duplication formula, we get

$$= \sum_{n=0}^{\infty} \frac{(e)_n (2xt)^n}{n!\, c^{\nu+n}} \sum_{k=0}^{\infty} \frac{\left(\frac{e+n}{2}\right)_k \left(\frac{e+n+1}{2}\right)_k (\nu+n)_k}{k! \left(\frac{\nu+n}{2}\right)_k \left(\frac{\nu+n+1}{2}\right)_k} \left[\frac{-(2x-y)^a t^2}{e}\right]^k$$

which is equivalent to (4.1).

Special Cases

(I) For y = c = 1, a = 0, then (2.1) and (2.2) reduces to

[8; p.57 (2.10 & 2.11)]

For c = 1, a = 0, ν = 1/2, then (2.1) and (2.2) reduces to

[3; p.164 (1)]

(II) For a = 0, c = 1 then (3.1) gives hypergeometric representation of Gegenbauer polynomial, i.e.

$$C_n^{\nu}(x) = \frac{(\nu)_n (2x)^n}{n!} \, {}_2F_1\left[\begin{matrix} -\frac{n}{2}, \frac{-n+1}{2}; \\ 1-\nu-n; \end{matrix} \; \frac{1}{x^2}\right] \qquad \ldots(5.1)$$

which is a known result [3; p.280 (19)].

For ν = 1/2 in (5.1), we get [3; p.166(4)].

(III) For c = a = y = 1 in (4.1), we get

$$\sum_{n=0}^{\infty} \frac{(e)_n S_n^{\nu}(x) t^n}{(\nu)_n} = \sum_{n=0}^{\infty} \frac{(e)_n (2xt)^n}{n!}$$

$$\times \, {}_3F_2\left[\begin{matrix} \frac{e+n}{2}, \frac{e+n+1}{2}, \nu+n; \\ \frac{\nu+n}{2}, \frac{\nu+n+1}{2}; \end{matrix} \; -(2x-1)t^2\right], \qquad \ldots(5.2)$$

which is a known generating function for $S_n^{\nu}(x)$ [8; p.60(4.6)].

For e = ν, then (5.2) gives

$$\sum_{n=0}^{\infty} S_n^{\nu}(x)t^n = \sum_{n=0}^{\infty} \frac{(\nu)_n (2xt)^n}{n!} {}_1F_0\left[\begin{matrix} \nu+n; \\ -\ ; \end{matrix} -(2x-1)t^2\right], \qquad \ldots(5.3)$$

which is a known result given by Sinha [13; p.439(2)].

(IV) For y = 0 in (4.1), we get

$$\sum_{n=0}^{\infty} \frac{(e)_n \phi_n^{\nu}(x)t^n}{(\nu)_n}$$

$$= \frac{(e)_n (2xt)^n}{n!c^{\nu+n}} {}_3F_2\left[\begin{matrix} \frac{e+n}{2}, \frac{e+n+1}{2}, \nu+n\ ; \\ \frac{\nu+n}{2}, \frac{\nu+n+1}{2}; \end{matrix} \frac{-(2x)^a t^2}{c}\right], \qquad \ldots(5.4)$$

which is a new and un known result.

(V) For e = ν and y = 2x in (4.1), then we get

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x,2x)t^n = \sum_{n=0}^{\infty} \frac{(\nu)_n (2xt)^n}{n!c^{\nu+n}}$$

$$= c^{-\nu} {}_1F_0\left[\nu;-;\frac{2xt}{c}\right]. \qquad \ldots(5.5)$$

(VI) For e = ν in (4.1), we get

$$\sum_{n=0}^{\infty} \phi_n^{\nu}(x, y)t^n = \sum_{n=0}^{\infty} \frac{(\nu)_n (2xt)^n}{n!c^{\nu+n}} {}_1F_0\left[\nu+n\ ;-;\frac{-(2x-y)^n t^2}{c}\right] \qquad \ldots(5.6)$$

Equations (5.5) and (5.6) are new and un known results.

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