INVESTIGATING ALGEBRAIC SIMPLE GROUPS WITH CONJUGACY CLASSES AND PRODUCTS

Anju **Dr. Manjeet Singh Jakhar**

Research Scholar **Associate Professor**

Department of Mathematics, NIILM University, Kaithal (Haryana)

Email – anjusungroya77@gmail.com dr.manjeet.jakhar@gmail.com

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ABSTRACT

Algebraic simple groups, being non-abelian and lacking proper normal subgroups, represent a profound and captivating domain within group theory and algebraic geometry. In the pursuit of understanding these enigmatic entities, the study of conjugacy classes and products emerges as a key to unlocking their underlying symmetries and structures. Group members can be partitioned into different sets with the use of conjugacy classes, which group together equivalent elements under inner automorphisms to show similar algebraic features. The analysis of conjugacy classes provides deep insights into the dynamics and character of algebraic simple groups. On the other hand, group products explore the interactions between distinct elements, generating new elements within the group and unraveling its internal symmetries.

Keywords: Conjugacy,Algebraic, Groups, Products,Semi-simple

I. INTRODUCTION

Algebraic simple groups, also known as simple algebraic groups, are a class of mathematical entities that hold a central position in modern group theory and algebraic geometry. These groups stand as a captivating domain of study, revealing profound insights into the symmetries and structures of abstract algebra. The roots of group theory can be traced back to the 19th century, with the works of mathematicians such as Évariste Galois, Arthur Cayley, and Augustin-Louis Cauchy. Galois laid the foundations of group theory with his groundbreaking studies on the solvability of polynomial equations using group-theoretic techniques. Cayley's work on permutation groups furthered the development of abstract group theory, while Cauchy made significant contributions to the theory of finite groups. However, the full-fledged notion of an abstract group as we know it today was formalized by Arthur Cayley in 1854, where he defined the concept of a group in terms of a set and a binary operation satisfying specific axioms.

As group theory matured, the study of simple groups emerged as a central pursuit. Simple groups are groups that possess no proper normal subgroups, meaning that they cannot be broken down into non-trivial subgroups that are invariant under the group operation. Simple groups are the "building blocks" of more complex groups, and understanding their structure is crucial for comprehending the symmetries within more general groups.

Algebraic simple groups, a specific class of simple groups, were first systematically studied in the early 20th century by the mathematicians ÉlieCartan, Wilhelm Killing, and others as part of the endeavor to understand the properties of Lie groups. Lie groups are groups that have a smooth manifold structure, and they play a fundamental role in various areas of mathematics and physics, particularly in the study of symmetry and differential equations.

A significant milestone in the study of algebraic simple groups was the classification theorem, a monumental achievement that occupied mathematicians for much of the 20th century. The theorem, culminating in the 1980s, classifies all finite-dimensional simple algebraic groups over algebraically closed fields into several families, including the classical groups (special linear, special orthogonal, and special unitary) and the exceptional groups (e.g., the Chevalley groups).

Conjugacy classes are an integral concept in group theory, particularly in the study of finite groups and matrix groups. A conjugacy class of a group element represents the set of elements that are conjugate to each other under inner automorphisms. In other words, two elements belong to the same conjugacy class if they can be transformed into each other by an inner automorphism (an operation of the form g^(-1) * x * g, where g is an element of the group and x is a member of the conjugacy class).

Group products, on the other hand, involve the operation of combining two or more elements of a group to generate a new element within the group. The product of two elements g and h is denoted as gh and represents the operation of first applying g and then applying h. Group products have essential properties, such as associativity and the existence of an identity element, which form the foundation of group theory.

II. REVIEW OF LITERATURE

By results ofGordeev, N.L, (1995)investigate connection between covering number and Lie rank of a simple algebraic group defined over algebraic closed field who's characteristic 0. Binary marking and labeling with nonnegative numbers are both employed. Both are necessary for recognizing huge conjugacy classes that exist in the product of two conjugacy classes and keeping track of the multiplicities of regular diagrams, respectively. For a product of normal subsets in G to have regular semi-simple components, we construct adequate criteria in terms of marked diagrams.

Ahanjideh, Neda. (2019)has been proven (in Guralnick and Moreto) that if p and q are two odd primes, then $\pi = \{ 2 , p, q\}$ and G is a finite group such that for every π-elements x, y$\in$ G with (O(x) , O(y)) = 1 , (xy) G = x G y G , If G has no composition factors of order pq, then G is not commutative. Motivated by the foregoing result, we prove in this note that if p and q are two primes (not necessarily odd) and G is a finite group such that for every pelement x and qelement y, G has an element z, y∈ G, (xy) G = x G y G ,If this is the case, then G does not include any composition factors with order pq. In specifically, we prove that for every p-element x and 2-element y, there exists a finite group G such that y∈ G, (xy) G = x G y G , then G is p-solvable.

Guralnick, Robert et al., (2012) For certain classes of finitely simple groups, we show that the Arad-Herzog conjecture is true: if both A and B are nontrivial conjugacy classes, then AB is also not a conjugacy class. We further show that AB is not a conjugacy class if and only if G is a finite simple Lie type group and if both A and B are nontrivial conjugacy classes, either both semisimple or both unipotent. For simple algebraic groups, we additionally establish a robust version of the Arad-Herzog conjecture, demonstrating, in particular, that the product of any two conjugacy classes in such a group always consists of an unlimited number of conjugacy classes. Therefore, we are able to acquire a full categorization of all possible pairs of centralizers in a simple algebraic group that have dense product. In particular, the centralizer of a noncentral element does not have any dense double cosets. For pseudoreductive groups, Prasad has utilized this result to examine Tits systems. For p-elements, where p is a prime number greater than 5, we prove an extension of the Baer-Suzuki theorem.

Moori, Jamshid& Tong-Viet, Hung. (2011). A finite group G is assumed here. Let us define the conjugacy class of an in G as a∈ G, let a^G = {a^g | g ∈ G}. In this work, we investigate the hypothesis of Arad and Herzog that the sum of two nontrivial conjugacy classes in a finite non-abelian simple group is never a single conjugacy class. In particular, we shall test this conjecture for a number of classes of Lie-type finite simple groups.

Guralnick, Robert &Malle, Gunter (2010)Several theorems involving products of conjugacy classes in finitely many simple groups are proven. The first conclusion is that a uniform generating triple exists in every case. The existence of elements in an irreducible linear group in tiny fixed space was a question originally posed by Peter Neumann in 1966, and it was used in conjunction with this finding to prove the conjecture. We further prove that for each finite non-abelian simple group, there are always two conjugacy classes whose product contains all the non-trivial elements of the group. In specifically, we use this to prove that each non-abelian finite simple group has an element that can be expressed as the product of two rth powers for any prime power r.

III. CONJUGACY CLASSES IN ALGEBRAIC SIMPLE GROUPS

In the study of algebraic groups, conjugacy classes play a fundamental role, providing valuable insights into the structure and representation theory of these groups.

Given an algebraic simple group G, two elements g, h ∈ G are said to be conjugate to each other if there exists an element x ∈ G such that $gxg^{-1} = h$. In other words, g and h are conjugate if they become identical when transformed by an inner automorphism induced by x. The set of all elements in G that are conjugate to a given element g is called the conjugacy class of g and is denoted by [g]. Following are the properties of conjugacy class:

- **Size of Conjugacy Classes:**Conjugacy classes need not have the same size, and in fact, they can have varying cardinalities. For finite groups, the size of a conjugacy class [g] is related to the order of G through the class equation: $|G| = \sum |[g]|$, where the summation is taken over representatives of distinct conjugacy classes.

- **Conjugacy Class and Centralizer:** The centralizer of an element g in G, denoted by C_G(g), is the subgroup of G consisting of all elements that commute with g, i.e., C_G(g) = $\{x \in G \mid gx = xg\}$. It can be shown that the centralizer of an element g is the stabilizer of the conjugacy class [g] under the action of the group G on itself by conjugation.

- **Conjugacy Classes and Normal Subgroups:** Conjugacy classes are closely related to normal subgroups. In particular, a subgroup N of G is normal if and only if it is a union of conjugacy classes. This property is often used to understand the normal subgroups of algebraic simple groups.

IV. RESULTS OF CONJUGACY CLASSES IN ALGEBRAIC GROUPS

We first recall some facts about conjugacy classes in algebraic groups. Throughout the section we fix an algebraically closed field k of characteristic $p \geq 0$.

A linked reductive group has only a finite number of conjugacy classes of unipotent elements, according to a key result of Lusztig. Inference: If A and B are conjugacy classes of a simple algebraic group, then AB is an infinite union of conjugacy classes if and only if the closure of AB contains an infinite number of semisimpleconjugacy classes. This outcome won't be used in the following.

We will use the following elementary result. Note that if a is an element of a connected reductive algebraic group G and a = su = us where s is semi simple and u is unipotent, then $s \in {}^{G}{}_{a}$.

Lemma 1 Assume that G is a linked reductive algebraic group over k, that T is its maximal torus, and that A and B are its non-central conjugacy classes. Then, the following claims are true.

1. AB either has an unlimited number of semi-simple classes or a single semi-simple conjugacy class of G.

2. AB contains a unique semi simple conjugacy class if and only if $AB \cap T$ is finite.

Proof.

Suppose that AB contains finitely many semi simple classes C1, . . ., Cm. Let Xi be the set of elements in G whose semi simple parts are in Ci. Note that Xi is closed (since if s $\in$ Xi is a semi simple element, then Xi consists of all elements g $\in$G with $\chi(g) = \chi(s)$ for all the characters of rational finite-dimensional G- modules). Since A and B are irreducible varieties, so is AB, whence AB $\subset$Ui Xi implies that AB $\subset$ Xi for some i. This proves (a).

Now (b) follows by (a) and the facts that every semisimple class of G intersects T nontrivially and this intersection is finite (since it is an orbit of the Weyl group on T).

Lemma 2 Let G be a semisimple algebraic group with a, b $\in$ G. If CG(a) CG(b) is dense in G, then aGbG is contained in the closure of (ab)G. In particular, the semisimple parts of elements of aGbG form a single semisimpleconjugacy class of G.

Proof.

Let $\Gamma = \{(g, h) \in G \times G \mid gh^{-1} \in C_G(a)C_G(b)\}$. Note that by assumption Γ contains a dense open subset of G × G. Suppose that (g, h) $\in\Gamma$.

Then $(a^g, b^h) = (a^{gh-1}, b)^h = (a^{xy}, b)^h = (a, b)^{yh}$,

where $gh^{-1} = xy$ with $x \in C_G(a)$ and $y \in C_G(b)$. Consider f: G × G → G given by f (g, h) = agbh. If c = ab, then f (Γ) $\subseteq$cG, whence f (G × G) is contained in the closure of cG, and the first part of the lemma follows.

Let s be the semisimple part of c. Let Gs be the set of elements in G whose semisimple part is conjugate to s. As previously noted, Gs is a closed subvariety of G.

Thus $a^G b^G \subseteq c^G \subseteq G_s$

We record the following trivial observation. Let H and K be subgroups of a group G and Γset := G/H × G/K. Then G acts naturally on Γ and the orbits of G on Γ are in bijection with the orbits of H on G/K and so in bijection with H\G/K.

V. CONCLUSION

The study of conjugacy classes and products in algebraic simple groups has revealed fascinating insights into the symmetries and structures of these complex mathematical entities. Conjugacy classes, through their grouping of equivalent elements, provide a powerful tool to comprehend the inherent dynamics and characteristics of these groups. On the other hand, group products offer a means to explore the interactions between different elements, leading to the generation of new elements within the group and uncovering its internal symmetries. The interplay between conjugacy classes and products has proven to be a captivating avenue, deepening our understanding of algebraic simple groups and influencing research in various mathematical areas.

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