"Generalized 2-Norms on Menger Linear Spaces over Menger Fields: A Comprehensive Study"

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Abstract: This article introduces a generalization of the concept of a 2-norm on Menger spaces over Menger fields, extending foundational principles of norm theory to these algebraic structures. The study outlines fundamental axioms governing the 2-norm in this context, establishing a framework for analysing vector magnitudes and distances within Menger spaces. By adapting notions from functional analysis and linear algebra to the setting of Menger spaces and fields, the research seeks to deepen understanding of geometric properties and algebraic structures in these generalized spaces. The exploration of 2-norms in Menger settings offers insights into the interplay between metric properties and algebraic structures, paving the way for further developments in mathematical theory and applications within this framework.

Keywords*:Menger Field, Menger Linear Space, 2-Norm on Menger Linear Spaces*

1. INTRODUCTION: The notion of a 2-norm on Menger linear spaces over Menger fields draws inspiration from L.A. Zadeh's seminal work on 2-norms in fuzzy linear spaces over fuzzy fields in 1965. Building upon Zadeh's framework, which revolutionized the understanding of vector spaces in the context of fuzzy logic, this article extends the concept to Menger spaces and fields. Karl Menger's introduction of Menger topology further motivated researchers to explore geometric and algebraic structures in generalized spaces. Since then, various scholars have contributed to the development of Menger norm theory from diverse perspectives. The Menger norm, emerging at the intersection of functional analysis, linear algebra, and Menger topology, serves as a fundamental tool for quantifying vector magnitudes and distances within Menger spaces. By generalizing the notion of a 2-norm to Menger linear spaces over Menger fields, this research aims to deepen our understanding of geometric properties and algebraic structures in these generalized settings. Such endeavors not only enrich theoretical mathematics but also hold promise for practical applications across diverse domains**.**

2. PRELIMINARIES In this section, we provide essential definitions and preliminary results crucial for establishing the proofs of our main results.

Definition 2.1. Let X be a field of a Menger space and F is a Menger set in X with the following Axioms

(a) $F_{(x+y)} \geq min\{,F_{(x)},F_{(y)}\}$, $\forall\, x,y \in X,$

(b) $F_{(-x)} \geq F_{(x)}$, $\forall\, x, \in X,$

(c) $F_{(xy)} \geq min\{,F_{(x)},F_{(y)}\}$, $\forall\, x,y \in X,$

(d) $F_{(x^{-1})} \geq F_{(x)}$, $\forall\, (x \neq 0), \in X.$

Theorem 2.1. *If*(F,X) is a Menger Field of X, then

(a) $F_{(0)} \geq F_{(x)}$, $\forall\, x, \in X,$

(b) $F_{(1)} \geq F_{(x)}$, $\forall\, (x \neq 0), \in X,$

(c) $F_{(0)} \geq F_{(1)}$.

Proof: we have

(a) $F_{(x-x)} \geq min\{,F_{(x)},F_{(-x)}\}$, $\forall\, x,y \in X,$

$$F_{(0)} \geq F_{(x)} \ , \qquad \forall\, x, \in X.$$

(b) $F_{(xx^{-1})} \geq min\{F_{(x)},F_{(x^{-1}}\}$

$$F_{(1)} \geq F_{(x)} \ , \qquad \forall\, (x \neq 0), \in X,$$

(c) $F_{(1-1)} \geq min\{,F_{(1)},F_{(-1)}\}$, $\forall\, x,y \in X,$

$$F_{(0)} \geq F_{(1)} \ .$$

Definition 2.2. Let X be a Field and (F,X) be a Menger Field of X.let Y be a linear space over X and V is a Menger set of Y.suppose the following condition satisfies

(i) $V_{(x+y)} \geq min\{V_x,V_y\}$, $x,y\epsilon Y,$
(ii) $V_{(-x)} \geq V_{(x)}$, $\forall\, x, \in Y,$
(iii) $V_{(\lambda x)} \geq min\{F_\lambda,V_{(x)}\}$, $\lambda\epsilon X, x\epsilon Y,$
(iv) $F_{(1)} \geq V_{(0)}$.
Then (V,Y) is called a Menger linear space over (F,X).

Theorem 2.2.*If*(V,Y) is a Menger linear space over (F,X). then

(i) $F_{(0)} = V_{(0).}$
(ii) $V_{(0)} = V_{(x)}, x \epsilon Y.$
(iii) $F_{(0)} = V_{(x)}, x \epsilon Y$

Proof:

(i) $F_{(0)} = F_{(1-1)} \geq F_{(1)} \geq V_{(0)}$
$F_{(0)} \geq V_{(0)}.$
(ii) $V_{(x-x)} \geq min\{V_{(x)}, V_{(-x)}\} = V_{(x)}$
$V_{(0)} = V_{(x)}, x \epsilon Y$.

(iii) $F_{(0)} = V_{(x)}, x \epsilon Y$ by (i) and (ii)

Theorem 2.3. Let (F, X) be a Menger Field of x and let$(V_1, Y_1), (V_2, Y_2), (V_3, Y_3) \dots . (V_n, Y_n)$ be Menger linear space over(F, X). Then$V_1 \times V_2 \times V_3 \times V_4 \dots \times V_n$,$Y_1 \times Y_2 \times Y_3 \times Y_4 \dots \times Y_n$ is a Menger linear space over(F, X).

Definition 2.4.Let (F, K)or $(F, R) or (F, C)$ be a Menger field of K , K denotes either R is the (set of real numbers) or C is the (set of complex numbers),F be a linear space over (V, X) be Menger linear space over (F, K),A norm on (V, X) is a function $\|.\|: X \to [0, \infty]$ such that

(i) $F_{\|x\|} \geq V_{(x)}, \forall\, x \epsilon X,$
(ii) $\|x\| \geq 0$ $, \forall\, x \epsilon X \& \|x\| = 0, if_{if} x = 0$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$ $, \forall\, x, y \epsilon X,$
(iv) $\|kx\| = |k| \|x\|$ $, \forall\, x, \epsilon X \& k \epsilon K$
Then $(V, X, \|.\|)$ is called a normal Menger linear space (NMLS).

3. Main Result: 2-Norm on Menger Linear spaces

Here K denotes either R is the (set of real numbers) or C is the (set of complex numbers)

Definition 3.1. Let(F, K) is a Menger field of K, X be a linear space over K and (V, X) be Menger linear space over (F, K) a 2-norm on (V, X) is a function$\|.\|: X \times X \to [0, \infty)$ such that

(i) $F_{\|x,y\|} \geq min\{V_{(x)}, V_{(y)}\}, \forall\, x, y \epsilon X,$
(ii) $\|x, = y\| = 0 \Leftrightarrow x \& y are L.I,$
(iii) $\|x, y\| = \|y, x\|,$
(iv) $\|x + y\, , z\| \leq \|x, z\| + \|y, z\|,$
(v) $\|kx, y\| \leq |k| \|x, y\|, \forall\, x, y \epsilon X, \& k \epsilon K.$

Then $(V, X, \|.\|)$ is called a 2-Normal Menger linear space on(F, R).

Proof: Suppose here$X = R$and $V = F$,then

$$F_{\|x,y\|} \geq min\{V_{(x)}, V_{(y)}\}\ ,$$

If x and y are linearly dependent then $F_{\|x,y\|} = F_{(0)} \geq F_{(x)}$,$\forall\ x\epsilon R$

If x and y are linearly independent then$F_{\|x,y\|} \geq\ min\{F_{(x)}, F_{(y)}\}$,$\forall\ x\epsilon R$.

(F, R)be a Menger field of R the 2-Norm $\|.\|$ on R^n define by $\|x, y\| = \|x_1, y_1\| + \quad \|x_2, y_2\| \dots. + \|x_n, y_n\|$is 2-Norm on the Menger linear space $(F \times F \times F \dots \times F, R^n)$

Now Let $= (x_1, x_2, x_3 \dots \dots x_n)\epsilon R^n$,$y = (y_1, y_2, y_3 \dots \dots y_n)\epsilon R^n$

$$F_{(\|x,y\|)} = F_{\{\|x_1,y_1\|+,\dots.+\|x_n,y_n\|\}}$$

$$\geq min\left\{F_{\{\|x_1,y_1\|,\dots.\|x_n,y_n\|\}}\right\}$$

$$\geq min\left\{F_{(x_1),}F_{(y_1)}\right\} \dots \dots. min\left\{F_{x_n}, F_{y_n}\right\}$$

$$= min\left\{(F \times F \times F \dots \times F)_{(x_1,x_2,x_3\dots..x_n)}, (F \times F \times F \dots \times F)_{((y_1,y_2,y_3\dots..y_n))}\right\}$$

$= \left\{{F_{(x)}}^n, {F_{(y)}}^n\right\}$.

Again Let(F, R) be a Menger field of R then the 2-Norm$\|.\| on R^n$ defined by

$\|x, y\| = min\left\{\{\|x_1, y_1\|, \dots. \|x_n, y_n\|\}\right\}$is 2-norm on the Menger linear space

$$(F \times F \times F \dots \times F, R^n).$$

Conclusion: In conclusion, this article presents a generalization of Zadeh's work on 2-norms in fuzzy linear spaces over fuzzy fields, extending it to Menger linear spaces over Menger fields. The result offers a versatile framework applicable to 2-norm Menger linear spaces over Menger fields. By leveraging foundational concepts from Menger topology and norm theory, this generalization facilitates deeper insights into geometric and algebraic structures within these generalized spaces. Such advancements not only contribute to theoretical mathematics but also hold practical relevance across diverse applications**.**

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