Ulam-Hyers Stability of Quadratic FunctionalEquation in Banach Space

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Abstract: In this paper, we introduce a new quadratic functional equation of three variable,and examine its Hyers-Ulam stability of this functional equation in Banach space using direct and fixed point method.

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1. Introduction:

The study of stability involves the use of functional equations. Ulam [14] proposed stability concerns of functional equations involving group homomorphisms in 1940. Under the presumption that groups are Banach spaces, Hyers [8] responded positively to Ulam's query regarding additive groups in 1941.Aoki [2] and Rassias[12] extended Hyers' theorem to include additive mappings and linear mappings, respectively, by taking into account an unbounded Cauchy difference $||\phi(v+y)-\phi(v)-\phi(y)|| \leq \varepsilon(||v||^p+||y||^p)$ for all $\varepsilon > 0$ and $p \in [0,1)$. Gavruta [5] also presented Rassias generalization theorem, substituting a control function $\varphi(v,y)$ for $\varepsilon(||v||^p+||y||^p)$. The concept of the Hyers-Ulam-Rassias stability of functional equations has been developed largely thanks to Rassias' publication. In 1982, Rassias [13] adopted the Rassias theorem [14]'s contemporary methodology, substituting the factor product of norms for the sum of norms.

Hyer's theorem has been expanded in a number of ways over the past few decades; for a list see ([1], [3], [4], [6], [7], [9], [10], [11]) The present work introduces a quadratic functional equation follow as:

$$\phi(x+y-2z)+\phi(x-2y+z)=\phi(2y-2z)+\phi(x-z)+\phi(x-y) \quad (1)$$

and derive its solution. Also, obtains Hyer- Ulam-Rassias stability in Banach space.

2. General Solution

Theorem 2.1. If a mapping $\phi: E \to F$ satisfying the functional equation (1), then the mapping $\phi: E \to F$ is quadratic.

Proof: Putting $\mathrm{x} = \mathrm{y}$ and $z = 0$ in equation (1), we get

$$\phi(2y) + \phi(-\mathrm{y}) = \phi(2\mathrm{y}) + \phi(\mathrm{y}) + \phi(0) \quad (2)$$

$$\phi(-\mathrm{y}) = \phi(y) + \phi(0) \quad (3)$$

Taking $\mathrm{x} = \mathrm{y} = z$ in equation (1) it will be $\phi(0) = 0$.

Then equation (3) becomes

$$\phi(-\mathrm{y}) = \phi(\mathrm{y}) \quad (4)$$

Taking $z = 0$ in equation (1), we get

$$\phi(x + \mathrm{y}) + \phi(\mathrm{x} - 2\mathrm{y}) = \phi(2\mathrm{y}) + \phi(\mathrm{x}) + \phi(\mathrm{x} - \mathrm{y})$$

$$\phi(\mathrm{x} + \mathrm{y}) = \phi(2\mathrm{y}) + \phi(\mathrm{x}) + \phi(\mathrm{x} - \mathrm{y}) - \phi(\mathrm{x} - 2y) \quad (5)$$

Similarly, taking $z = 0$ and $y = -y$ in equation (1), we obtain

$$\phi(\mathrm{x} - y) = \phi(-2\mathrm{y}) + \phi(\mathrm{x}) + \phi(\mathrm{x} + \mathrm{y}) - \phi(\mathrm{x} + 2\mathrm{y}) \quad (6)$$

Adding equation (5) and equation (6), and using $\phi(-y) = \phi(\mathrm{y})$ we have

$$2\phi(2y) + 2\phi(\mathrm{x}) = \phi(\mathrm{x} - 2\mathrm{y}) + \phi(\mathrm{x} + 2\mathrm{y})$$

Now putting $\mathrm{y} = \frac{y}{2}$, we obtain

$$2\phi(x) + 2\phi(\mathrm{y}) = \phi(\mathrm{x} + \mathrm{y}) + \phi(\mathrm{x} - \mathrm{y}) \quad (7)$$

taking $\mathrm{x} = y$, in above equation, we get

$$\phi(2\mathrm{x}) = 2^2\phi(x) \quad (8)$$

clearly, this equation become a quadratic equation.

3. Stability of Quadratic Functional Equation

We define:

$$D(\mathrm{x},\mathrm{y},\mathrm{z}) = \phi(x+y-2z)+\phi(x-2y+z)-\phi(2y-2z)-\phi(x-z)-\phi(x-y)$$

(9)

for each $\mathrm{x},\mathrm{y},\mathrm{z} \in E$.

Theorem 3.1. Assume that V and W are Banach spaces. If a function $\phi: V \to W$ satisfies the inequality

$$||D\phi(x,y,z)|| < \varepsilon \quad (10)$$

for some $\varepsilon > 0$, for all $x, y, z \in V$, then the limit

$$Q_2(x) = \lim_{m\to\infty} \frac{\phi(3^m x)}{3^{2m}} \quad (11)$$

exists for each $x \in V$ and $Q_2: V \to W$ is unique quadratic function such that

$$||\phi(x) - Q_2(x)|| < \frac{\varepsilon}{8} \quad (12)$$

for any $x \in V$.

Proof: Replace (x, y, z) by $(z, 2z, 3z)$ in (10), we have

$$||\phi(3z) - 9\phi(z)|| < \varepsilon \quad (13)$$

$$\left\|\frac{\phi(3z)}{3^2} - \phi(z)\right\| < \frac{\varepsilon}{9} \quad (14)$$

Replace z by $3^{\mathrm{t}}z$ in (14), we have

$$\left\|\frac{\phi(3^{\mathrm{t}+1}z)}{3^2} - \phi(3^{\mathrm{t}}z)\right\| < \frac{\varepsilon}{9} \quad (15)$$

$$\left\|\frac{\phi(3^{\mathrm{t}+1}z)}{3^{2(\mathrm{t}+1)}} - \frac{\phi(3^t z)}{3^{2\mathrm{t}}}\right\| < \frac{\varepsilon}{3^{2(\mathrm{t}+1)}} \quad (16)$$

for all $z \in V$ and all $\varepsilon > 0$. Since

$$\frac{\phi(3^m z)}{3^{2m}} - \phi(z) = \sum_{i=0}^{m-1} \left(\frac{\phi(3^{i+1}z)}{3^{2(i+1)}} - \frac{\phi(3^i z)}{3^{2i}}\right) \quad (17)$$

So,

$$\left\|\frac{\phi(3^m z)}{3^{2m}} - \phi(z)\right\| \leq \sum_{i=0}^{m-1} \left\|\frac{\phi(3^{i+1}z)}{3^{2(i+1)}} - \frac{\phi(3^i z)}{3^{2i}}\right\| \tag{18}$$

$$< \sum_{i=0}^{m-1} \frac{\varepsilon}{3^{2(i+1)}} = \frac{\varepsilon}{8}\left(1 - \frac{1}{3^{2m}}\right). \tag{19}$$

Replace z by $3^{\mathrm{m}}z$, we get

$$\left\|\frac{\phi(3^{m+m}z)}{3^{2(m+m)}} - \frac{\phi(3^m z)}{3^{2m}}\right\| < \frac{\varepsilon}{8}\left(\frac{1}{3^{2m}} - \frac{1}{3^{2(m+m)}}\right), \tag{20}$$

for all $z \in V$ and all $\varepsilon > 0$. R.H.S $\to 0$ as $m \to \infty$ then $\left\{\frac{\phi(3^m z)}{3^{2m}}\right\}$ is a Cauchy sequence in W. Since W is Banach- space, thus sequence $\left\{\frac{\phi(3^m z)}{3^{2m}}\right\}$ converges to some $Q_2(z) \in W$. For $z \in V$,

$$||\mathrm{Q}_2(z) - \phi(z)|| = \left\|\mathrm{Q}_2(z) - \frac{\phi(3^{\mathrm{m}}z)}{3^{2\mathrm{m}}} + \frac{\phi(3^{\mathrm{m}}z)}{3^{2\mathrm{m}}} - \phi(z)\right\|$$

$$\leq \left\|\mathrm{Q}_2(z) - \frac{\phi(3^{\mathrm{m}}z)}{3^{2\mathrm{m}}}\right\| + \left\|\frac{\phi(3^{\mathrm{m}}z)}{3^{2\mathrm{m}}} - \phi(z)\right\|$$

$$< \left\|\mathrm{Q}_2(z) - \frac{\phi(3^{\mathrm{m}}z)}{3^{2\mathrm{m}}}\right\| + \frac{\varepsilon}{8}\left(1 - \frac{1}{3^{2\mathrm{m}}}\right). \tag{21}$$

for all $z \in V$ and all $\varepsilon > 0$. Taking the limit $m \to \infty$,

$$\|Q_2(z) - \phi(z)\| < \frac{\varepsilon}{8} \tag{22}$$

Replacing (x, y, z) by $(3^m x, 3^m \mathrm{y}, 3^m \mathrm{z})$ in (10), we have

$$\|D\phi(3^m \mathrm{x}, 3^m \mathrm{y}, 3^m \mathrm{z})\| < \varepsilon$$

$$\left\|D\phi\left(\frac{3^m \mathrm{x}}{3^{2m}}, \frac{3^m \mathrm{y}}{3^{2m}}, \frac{3^m \mathrm{z}}{3^{2m}}\right)\right\| < \frac{\varepsilon}{3^{2m}}. \tag{23}$$

Applying $m \to \infty$, show that Q_2 satisfies the functional equation (1).

To prove the uniqueness of Quadratic mapping Q_2. Assume that there exists another Quadratic mapping $Q_2^{'}$, which satisfies inequality (12). Fix $z \in V$. Clearly, $Q_2(3^t z) = 3^{2t} Q_2(z)$ and $Q_2^{'}(3^t z) = 3^{2t} Q_2^{'}(z)$ for all $z \in V$, from (12), we have

$$\|Q_2(z) - Q_2^{'}(z)\| = \left\|\frac{Q_2(3^m z)}{3^{2m}} - \frac{\phi(3^{2m}z)}{3^{2m}} + \frac{\phi(3^{2m}z)}{3^{2m}} - \frac{Q_2^{'}(3^m z)}{3^{2m}}\right\|$$

$$< \frac{1}{3^{2m-1}} \cdot \frac{\varepsilon}{8} \tag{24}$$

Taking $m \to \infty$, we have $Q_2(z) = Q'_2(z)$.

Theorem 3.2. Assume that V and W are Banach spaces. If a function $\phi: V \to W$ satisfies the inequality

$$||D\phi(x, y, z)|| \le \theta(||\mathrm{x}||^p + ||\mathrm{y}||^p + ||\mathrm{z}||^p). \tag{25}$$

For some $p < 3$, for all $x, y, z \in V$, then the limit

$$Q_2(z) = \lim_{m\to\infty} \frac{\phi(3^m z)}{3^{2m}} \tag{26}$$

exists for each $z \in V$ and $Q_2: V \to W$ is unique quadratic function such that

$$||\phi(z) - Q_2(z)|| \le \frac{\theta||z||^p}{(3^2 - 3^p)} \tag{27}$$

for all$z \in V$.

Proof: Replace (x, y, z) by $(z, 2z, 3z)$ in (25), we have

$$||\phi(3z) - 9\phi(z)|| \le \theta||z||^p \left\|\frac{\phi(3z)}{3^2} - \phi(z)\right\|$$

$$\le \frac{\theta||z||^{\mathrm{p}}}{3^2} \tag{28}$$

Replace z by $3^t z$ in (28), we have

$$\left\|\frac{\phi(3^{\mathrm{t}+1}z)}{3^2} - \phi(3^{\mathrm{t}}v)\right\| \le \frac{\theta||3^t z||^p}{3^2},$$

$$\left\|\frac{\phi(3^{t+1}z)}{3^{2(t+1)}} - \frac{\phi(3^t z)}{3^{2t}}\right\| \le \frac{\theta||z||^p}{3^{2(t+1)-tp}} \tag{29}$$

for all $z \in V$. Since

$$\frac{\phi(3^m z)}{3^{2m}} - \phi(z) = \sum_{i=0}^{m-1} \left(\frac{\phi(3^{i+1}z)}{3^{2(i+1)}} - \frac{\phi(3^i z)}{3^{2i}}\right) \text{ (30)}$$

So,

$$\left\|\frac{\phi(3^m z)}{3^{2m}} - \phi(z)\right\| \le \sum_{i=0}^{m-1} \left\|\frac{\phi(3^{i+1}z)}{3^{2(i+1)}} - \frac{\phi(3^i z)}{3^{2i}}\right\|$$

$$\leq \sum_{i=0}^{m-1} \frac{\theta||z||^p}{3^{2(i+1)-pi}}$$

$$= \frac{\theta||z||^p}{(3^2-3^p)}\left(1-\frac{1}{3^{m(2-p)}}\right) \tag{31}$$

Replace z by $3^m z$, we get

$$\left\|\frac{\phi(3^{m+m}z)}{3^{2(m+m)}}-\frac{\phi(3^m z)}{3^{2m}}\right\| \leq \frac{\theta||z||^p}{(3^2-3^p)}\left(\frac{1}{3^{2m}}-\frac{1}{3^{2(m+m)-mp}}\right) \tag{32}$$

for all $z \in V$. R.H.S $\rightarrow 0$ as $m \rightarrow \infty$ then $\left\{\frac{\phi(3^m z)}{3^{2m}}\right\}$ is a Cauchy sequence in W. Since W is Banach- space, thus sequence $\left\{\frac{\phi(3^m z)}{3^{2m}}\right\}$ converges to some $Q_2(z) \in W$. For $z \in V$,

$$\|Q_2(z)-\phi(z)\| = \left\|Q_2(z)-\frac{\phi(3^m z)}{3^{2m}}+\frac{\phi(3^m z)}{3^{2m}}-\phi(z)\right\|$$

$$\leq \left\|Q_2(z)-\frac{\phi(3^m z)}{3^{2m}}\right\| + \left\|\frac{\phi(3^m z)}{3^{2m}}-\phi(z)\right\|$$

$$\leq \left\|Q_2(z)-\frac{\phi(3^m z)}{3^{2m}}\right\| + \frac{\theta||z||^p}{(3^2-3^p)}\left(1-\frac{1}{3^{m(2-p)}}\right), \tag{33}$$

for all $z \in V$. Taking the limit $m \rightarrow \infty$,

$$\|Q_2(z)-\phi(z)\| \leq \frac{\theta||z||^p}{(3^2-3^p)} \tag{34}$$

Replacing (x, y, z) by $(3^m x, 3^m y, 3^m z)$ in (25), we have

$$\|D\phi(3^m x, 3^m y, 3^m z)\| \leq \theta(||3^m x||^p + ||3^m y||^p + ||3^m z||^p)$$

$$\left\|D\phi\left(\frac{3^m x}{3^{2m}}, \frac{3^m y}{3^{2m}}, \frac{3^m z}{3^{2m}}\right)\right\| \leq \frac{\theta}{3^{2m-mp}}(||3^m x||^p + ||3^m y||^p + ||3^m z||^p) \tag{35}$$

Applying $m \rightarrow \infty$, show that Q_2 satisfies the functional equation (1).

To prove the uniqueness of Quadratic mapping Q_2. Assume that there exists another Quadratic mapping Q_2', which satisfies inequality (27). Fix z ∈ V. Clearly, $Q_2(3^t z) = 3^{2t} Q_2(z)$ and $Q_2'(3^t z) = 3^{2t} Q_2'(z)$ for all $z \in V$. We have

$$\|Q_2(z)-Q_2'(z)\| = \left\|\frac{Q_2(3^m z)}{3^{2m}}-\frac{\phi(3^m z)}{3^{2m}}+\frac{\phi(3^m z)}{3^{2m}}-\frac{Q_2'(3^m z)}{3^{2m}}\right\|$$

$$\leq \frac{\theta||z||^p}{3^{2m-mp-1}(3^2-3^p)} \tag{36}$$

Taking $m \to \infty$, we have $Q_2(z) = Q'_2(z)$.

Theorem 3.3. Assume that V and W are Banach spaces. Let $\varphi: V^m \to R^+$ be a function such that $\sum_{i=0}^{\infty} \frac{\varphi(3^i z,\ 3^i 2z,\ 3^i 3z)}{3^{2i}}$ converges and $\lim_{i\to\infty} \frac{\varphi(3^i z,\ 3^i 2z,\ 3^i 3z)}{3^{2i}} = 0$. Also, if a function $\phi: V \to W$ satisfies the inequality

$$\|D\phi(x,y,z)\| \le \varphi(\mathrm{x},\mathrm{y},\mathrm{z}) \tag{37}$$

for all $x, y, z \in V$, then the limit $Q_2(z) = \lim_{m\to\infty} \frac{\phi(3^m z)}{3^{2m}}$, exists for each z in Vand $Q_2: V \to W$ is unique quadratic function such that

$$\|\phi(z) - Q_2(z)\| \le \sum_{i=0}^{\infty} \frac{\varphi(3^i z,\ 3^i 2z,\ 3^i 3z)}{3^{2(i+1)}}. \tag{38}$$

for any $z \in V$.

Proof: Replace (x, y, z) by $(z, 2z, 3z)$ in (37), we have

$$\|\phi(3\mathrm{z}) - 3^2\phi(z)\| \le \varphi(z, 2z, 3z)$$

$$\left\|\frac{\phi(3z)}{3^2} - \phi(z)\right\| \le \frac{\varphi(z,\ 2z,\ 3z)}{3^2}. \tag{39}$$

Replace z by $3^t z$ in (39), we have

$$\left\|\frac{\phi(3^{\mathrm{t}+1} z)}{3^2} - \phi(3^{\mathrm{t}} z)\right\| \le \frac{\varphi(3^t z,\ 3^t 2z,\ 3^t 3z)}{3^2},$$

$$\left\|\frac{\phi(3^{t+1} z)}{3^{2(t+1)}} - \frac{\phi(3^t z)}{3^{2t}}\right\| \le \frac{\varphi(3^t z,\ 3^t 2z,\ 3^t 3z)}{3^{2(t+1)}} \tag{40}$$

for all $z \in V$. Since

$$\frac{\phi(3^m z)}{3^{2m}} - \phi(z) = \sum_{i=0}^{m-1} \left(\frac{\phi(3^{i+1} z)}{3^{2(i+1)}} - \frac{\phi(3^i z)}{3^{2i}}\right) \quad (41)$$

So,

$$\left\|\frac{\phi(3^m z)}{3^{2m}} - \phi(z)\right\| \le \sum_{i=0}^{m-1} \left\|\frac{\phi(3^{i+1} z)}{3^{2(i+1)}} - \frac{\phi(3^i z)}{3^{2i}}\right\|$$

$$\le \sum_{i=0}^{m-1} \frac{\varphi(3^t z,\ 3^t 2z,\ 3^t 3z)}{3^{2(t+1)}}. \tag{42}$$

Replacing z by $3^m z$ in (42), we get

$$\left\|\frac{\phi(3^{m+m}z)}{3^{2(m+m)}}-\frac{\phi(3^m z)}{3^{2m}}\right\| \leq \sum_{i=0}^{m+m-1} \frac{\varphi(3^t z,\ 3^t 2z,\ 3^t 3z)}{3^{2(2+i)}}+\sum_{i=0}^{m-1} \frac{\varphi(3^t z,\ 3^t 2z,\ 3^t 3z)}{3^{2(i+1)}}$$

$$\leq \sum_{i=m}^{m+m-1} \frac{\varphi(3^t z,\ 3^t 2z,\ 3^t 3z)}{3^{2(i+1)}} \tag{43}$$

for all $z \in V$.

Taking the limit $m \to \infty$, we have

$$\|Q_2(z)-\phi(z)\| \leq \sum_{i=0}^{\infty}, \frac{\varphi(3^t z,\ 3^t 2z,\ 3^t 3z)}{3^{2(i+1)}} \tag{44}$$

Replacing (x, y, z) by $(3^m x, 3^m y, 3^m z)$ in (37), we have

$$\|D\phi(3^m x, 3^m y, 3^m z)\| \leq \varphi(3^m x, 3^m y, 3^m z)$$

$$\left\|D\phi\left(\frac{3^m x}{3^{2m}}, \frac{3^m y}{3^{2m}}, \frac{3^m z}{3^{2m}}\right)\right\| \leq \frac{\varphi(3^m x,\ 3^m y,\ 3^m z)}{3^{2m}}. \tag{45}$$

Applying $m \to \infty$, show that Q_2 satisfies the functional equation (1).

To prove the uniqueness of Quadratic mapping Q_2. Assume that there exists another Quadratic mapping Q_2', which satisfies inequality (38). Fix $z \in V$. Clearly, $Q_2(3^t z) = 3^{2t} Q_2(z)$ and $Q_2'(3^t z) = 3^{2t} Q_2'(z)$ for all $z \in V$. We have

$$\|Q_2(z)-Q_2'(z)\| = \left\|\frac{Q_2(3^m z)}{3^{2m}}-\frac{\phi(3^m z)}{3^{2m}}+\frac{\phi(3^m z)}{3^{2m}}-\frac{Q_2'(3^m z)}{3^{2m}}\right\|$$

$$\leq \sum_{i=m}^{\infty} \frac{\varphi(3^i z,\ 3^i 2z,\ 3^i 3z)}{3^{2(+1)}}+\sum_{i=m}^{\infty} \frac{\varphi(3^i z,\ 3^i 2z,\ 3^i 3z)}{3^{2(i+1)}} \tag{46}$$

Taking $m \to \infty$, we have $Q_2(z) = Q_2'(z)$.

4. Stability of Functional Equation (1) using Fixed Point Method

Theorem: C (Banach contraction principle) Let (V, d) be a complete metric spaces consider a mapping $T: V \to V$ which is strictly contractive mapping, that is,

(C_1) $d(Tz, Ty) \leq d(z, y)$ for some (Lipschitz constant) $L < 1$, then

(i) The mapping T only has one fixed point, which is $T(z*) = z*$.

(ii) Every given element has a fixed point (z*) that is universally contractive.

(C_2) $\lim_{m\to\infty} T^m z = z*$ for any starting point $z \in V$.

(iii) One has the following estimation inequalities,

(C_3) $d(T^m z, z*) \leq \frac{1}{1-L} d(T^m z, T^{m+1} z), \forall\, m \geq 0$, for all $z \in V$.

(C_4) $d(z, z*) \leq \frac{1}{1-L} d(z, z*)$ for all $z \in V$.

Theorem: D (Alternative Fixed Point)

Itf a generalized metric space (V, d) is complete and a strictly contractive mapping $T: V \to V$has a Lipschitz constant L, then for any given element $z \in V$ either,

(D_1) $d(T^m z, T^{m+1} z) = \infty\ \forall\, m \geq 0$.

(D_2) There exists a natural number such that,

(i) $d(T^m z, T^{m+1} z) < \infty\ \forall\, m \geq 0$.

(ii) The sequence $\{T^m z\}$ is convergent to a fixed point y^* of T.

(iii) y^* is the unique fixed point of T in the set $W = \{y \in W; d(T^{m_0}, y) < \infty\}$.

(iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$, for all $y \in W$.

Theorem4.1 Let $\phi: A \to B$ be an even mapping for which there exists a function $\varphi: A^m \to [0, \infty]$ with the condition

$\lim_{m\to\infty} \frac{\varphi(\xi_i^m \mathrm{x},\ \xi_i^m \mathrm{y},\ \xi_i^m \mathrm{z})}{\xi_i^m} = 0$ (47)

where,

$$\xi_i = \begin{cases} 3, & i = 1 \\ \frac{1}{3}, & i = 0 \end{cases}$$

such that the functional inequality

$$\|D\phi(x,y,z)\| \leq \varphi(\mathrm{x},\mathrm{y},\mathrm{z}) \tag{48}$$

for all $x, y, z \in V$. If there exists $L = L(i)$ such that the function

$$z \to \beta(z) = \varphi\left(\frac{z}{3}, \frac{2z}{3}, \frac{3z}{3}\right) \tag{49}$$

has the property,

$$\frac{1}{\xi_i}\beta(\xi_i z) = L\beta(z) \tag{50}$$

for each $z \in A$. Then there exists a unique quadratic mapping $Q_2: A \to B$ satisfying the functional equation (1) and

$$\|\phi(z) - Q_2(z)\| \leq \frac{L^{1-i}}{1-L}\beta(z) \tag{51}$$

holds for all $z \in A$.

Proof : Introduce the generalized metric to the set $V = \{P \backslash P: A \to B, P(0) = 0\}$ and then have a look at the set V. $d(p,q) = \inf\{K \in (0,\infty): ||p(z) - q(z)\| \leq K\beta(z),\ z \in A\}$. It is clear that (V,d) is complete. Define $T: V \to V$ by

$$T_p(z) = \frac{1}{\xi_i} p(\xi_i z) \tag{52}$$

for all $z \in A$. Now $p, q \in V$,

$$d(p,q) \leq K$$

$$\|p(z) - q(z)\| \leq K\beta(z),\ z \in A$$

$$\left\|\frac{1}{\xi_i}p(\xi_i z) - \frac{1}{\xi_i}q(\xi_i z)\right\| \leq \frac{1}{\xi_i}K\beta(\xi_i z),\ z \in A$$

$$\|Tp(z) - Tq(z)\| \leq LK\beta(z),\ z \in \mathrm{A}$$

$$d(Tp, Tq) \leq LK.$$

This means that $d(Tp,Tq) \leq Ld(p,q)$ for each $p,q \in V$. T is strictly contractive mapping on V with Lipschtiz constant L. It follows from (48) that

$$\|\phi(3z) - 9\phi(z)\| \leq \varphi(z, 2z, 3z) \quad (53)$$

for each $z \in A$. It follows from (53) that

$$\left\|\frac{\phi(3z)}{3^2} - \phi(z)\right\| \leq \frac{\varphi(z,\ 2z,\ 3z)}{3^2} \quad (54)$$

for each $z \in A$. From (50), for the case $i = 1$, it reduces to

$$\left\|\frac{\phi(3z)}{3^2} - \phi(z)\right\| \leq \frac{1}{8}\beta(z) \quad (55)$$

for each $z \in A$. (i.e.,) $d(\phi, T\phi) \leq \frac{1}{8} \Rightarrow d(\phi, T\phi) \leq \frac{1}{8} = L = L^{'} < \infty$. Again replace $z = \frac{z}{3}$ in (53), we obtain

$$\left\|\phi(z) - 9\phi\left(\frac{z}{3}\right)\right\| \leq \varphi\left(\frac{z}{3}, \frac{2z}{3}, \frac{3z}{3}\right) \quad (56)$$

for each $z \in A$. Using (50) for $i = 0$, it reduces to,

$$\left\|9\phi\left(\frac{z}{3}\right) - \phi(z)\right\| \leq \varphi(\beta(z)) \quad (57)$$

for each $z \in A$. (i.e.,) $d(\phi, T\phi) \leq 1 \Rightarrow d(\phi, T\phi) \leq 1 = L^0 < \infty$. In the above case we reached

$$d(\phi, T\phi) \leq L^{1-i} \quad (58)$$

Therefore $(\mathrm{C}_2(\mathrm{i}))$ hold. Using $(\mathrm{C}_2(\mathrm{ii}))$, it follows that exists a fixed point Q_2 of T in A, such that

$$Q_2(z) = \lim_{m\to\infty} \frac{\phi_a(\xi_i^m z)}{\xi_i^m}, \forall\ z \in A \quad (59)$$

To prove that $Q_2 \colon A \to B$ is quadratic. Using $(\xi_i^m x, \xi_i^m \mathrm{y}, \xi_i^m \mathrm{z})$ at place of (x,y,z) in (54) and dividing by ξ_i^m, it follows from (46) and (59), we see that Q_2 satisfies (1) for all $x,y,z \in V$. Hence Q_2 satisfies the functional equation (1).

By using $C_2(iii)$, Q_2 is the unique fixed point of T in the set, $W = \{\phi \in V; d(T\phi, Q_2) < \infty\}$. Using fixed point alternative result, Q_2 is the unique function such that,

$$\|\phi(z) - Q_2(z)\| \leq K\beta(z) \quad (60)$$

for all $z \in A$ and $k > 0$. Finally, by $(\mathrm{C}_2(\mathrm{iv}))$, we obtain

$$d(\phi, A) \leq \frac{1}{1-L} d(\phi, T\phi) \tag{61}$$

as $d(\phi, Q_2) \leq \frac{L^{1-i}}{1-L}$. Hence, we conclude that

$$\|\phi(z) - Q_2(z)\| \leq \frac{L^{1-i}}{1-L} \beta(z) \tag{62}$$

for each $z \in A$. This completes the proof.

5. Conclusion

In this manuscript a quadratic functional equation is invented. Hyers-Ulam-Rassias stability of this functional equation is proved in Banach space using two different method, one is direct method and another is fixed point method.

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