On Ulam Stability of Cubic Functional Equation in Random Normed Space

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Abstract: In this work, we will prove the Hyers-Ulam stability of cubic functional equation using direct and fixed point method in random normed space. Some literature results that directly follow our main results have also been demonstrated.

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1. Introduction

It is widely known that S.M. Ulam [22] raised the issue of the stability of homomorphisms of metric groups in 1940 during his lecture at the University of Wisconsin. D.H. Hyers [14] offered its solution in 1941 for the case of Banach spaces (let us note here that Ulam inquired about metric groups). Numerous researchers looked into this issue; for additional details on this idea, its findings, and its applications of Hyers-Ulam stability, the reader is directed to ([1], [3], [4], [7], [8], [11], [13], [17]). Afterwards, Aoki [2] expanded on Hyers's result in 1950 for additive mapping. For approximately linear mapping, Rassias ([5], [6]) offered a generalized version of Hyers in 1978. Additionally, Rassias introduced two weaker requirements, controlled by a mixed product-sum of powers of norms and a product of different powers of norms, respectively, to generalize the Hyers stability result.

The theory of random normed spaces, or RN - spaces, was first presented by Serstnev [21] in 1963. It is a generalization of the deterministic result of normed spaces and is closely related to the study of random operator equations. Numerous studies using various types of equations have been published on Random Normed Spaces (See [12], [15], [16], [18]). Fuzzy normed space that we are using to prove the stability of a cubic functional equation was explored by K. Ravi et al. (2011) [19]. The stability of the Cubic functional equation in random normed space will be covered in this study

$$\aleph(2\alpha+\beta)+\aleph(2\alpha-\beta)=2\aleph(\alpha+\beta)+2\aleph(\alpha-\beta)+12\aleph(\alpha) \tag{1}$$

Using the direct and fixed-point method in Random Normed space.

Because the function $\aleph(\alpha)=r\alpha^3$ is a solution of the aforementioned functional equation (1), functional equation (1) is also known as a cubic functional equation.

We require the following concepts and terminology from the literature in order to support our main results:

A function $\Re: R\cup\{-\infty,+\infty\}\to[0,1]$ with conditions nondecreasing, left -continuous, $\Re(0)=0$ and $\Re(\infty)=1$ is referred as a distribution function. A is the set of all probability distribution functions $\Re$ with $\Re(0)=0$. $D^+$is a subset of A consisting of all function $\Re\in A$ for which $\Re(\infty)=1$, where $l^-\Re(\alpha)=\lim_{\hbar\to\alpha^-}\Re(\hbar)$. For any $p\geq 0, \epsilon_p$ is the element of D^+which is defined as follow:

$$\epsilon_p\begin{cases}0\ if & \hbar\leq 0\\ 1\ if & \hbar>0\end{cases}$$

Definition 1 [21]A function $\Psi:[0,1]\times[0,1]\to[0,1]$ is a continuous triangular norm (briefly, a $\hbar$-norm) if Ψ satisfies the following conditions:

(1) Ψ is associative and commutative,

(2) Ψ is continuous,

(3) $\Psi(p,1)=1$ for all $p\in[0,1]$,

(4) $\Psi(p,q)\leq\Psi(r,s)$ whenever $p\leq r$ and $q\leq s$ for all $p,q,r,s\in[0,1]$.

The examples of continuous $\hbar$-norm are as follows:

$$\Psi_M(p,q)=\min\{p,q\},\Psi_P(p,q)=\min\{p,q\},\Psi_L(p,q)=\max\{p+q-1,0\}$$

Recall that, if Ψ is a $\hbar$-norm and $\{\alpha_n\}$ is a sequence of number in [0,1], then $\Psi_{i=1}^n\alpha_i$ is defined recurrently by $\Psi_{i=1}^1\alpha_i=\alpha_1$ and $\Psi_{i=1}^n\alpha_i=\Psi\left(\Psi_{i=1}^{n-1}\alpha_i,\alpha_n\right)=\Psi(\alpha_1,\alpha_2,\alpha_3,\ldots\alpha_n)$ for each $n\geq 2$ and $\Psi_{i=1}^\infty\alpha_n$ is defined as $\Psi_{i=1}^\infty\alpha_{n+i}$ [13].

Definition 2 [22] Let χ be a real linear space, κ be a mapping from χ into D^+(for any $\alpha\in\chi,\kappa(\alpha)$ is denoted by κ_α) and Ψ be a continuous norm. The triple $(\chi,\kappa,\hbar)$ is called a random normed space (briefly RN-space) if κ satisfies the following conditions:

$(RN1)$ $\kappa_\alpha(\hbar) = \epsilon_o(\hbar)$ for all $\hbar > 0$ if and only if $\alpha = 0$;

$(RN2)$ $\kappa_{a\alpha}(\hbar) = \kappa_\alpha\left(\frac{\hbar}{|a|}\right)$ for all $\alpha \in \chi, a \neq 0$ and all $\hbar \geq 0$;

$(RN3)$ $\kappa_{\alpha+\beta}(\hbar + s)(\hbar) \geq \Psi\big(\kappa_\alpha(\hbar), \kappa_\beta(s)\big)$ for all $\alpha, \beta \in \chi$ and all $\hbar, s > 0$.

Example 1 Every normed space $(\chi, \|\cdot\|)$ defines a RN-space (χ, κ, Ψ_M), where $\kappa_\alpha(\hbar) = \frac{\hbar}{\hbar+\|\alpha\|)}$, for all $\hbar > 0$ and Ψ_M is the minimum $\hbar$-norm. This space is called induced random normed space.

Definition 3 Let (χ, κ, Ψ) be RN-space

(1) If, for every $\hbar > 0$ and $\lambda > 0$, there exists a positive integer N such that $\kappa_{\alpha_n-\alpha}(\hbar) > 1 - \lambda$, whenever $n \geq N$, then a sequence $\{\alpha_n\}$ in χ is said to be convergent to a point α. As $\lim_{n\to\infty}\alpha_n = \alpha$, then α is known as the limit of the sequence $\{\alpha_n\}$.

(2) If, for every $\hbar > 0$ and $\lambda > 0$, there exists a positive integer N such that, whenever $n \geq m \geq M$, $\kappa_{\alpha_n-\alpha_m}(\hbar) > 1 - \lambda$, then the sequence $\{\alpha_n\}$ in χ is referred to as Cauchy sequence.

(3) If every Cauchy sequence in χ is convergent to a point in χ then the RN space (χ, κ, Ψ) is said to be complete .

Theorem 1 [21] If (χ, κ, Ψ) is a random normed space and $\{\alpha_n\}$ is a sequence of χ s.t. $\alpha_n \to \alpha$ then $\lim_{n\to\infty}\kappa_{\alpha_n}(\hbar) = \kappa_\alpha(\hbar)$ almost everywhere.

The general solution and generalized Hyers-Ulam stability of the cubic functional equation were demonstrated in 2011 by K. Ravi et al. [19]

$$\aleph(2\alpha + \beta) + \aleph(2\alpha - \beta) = 2\aleph(\alpha + \beta) + 2\aleph(\alpha - \beta) + 12\aleph(\alpha)$$

in fuzzy normed space.

In this work, χ will represent a real linear space, (Z, κ', Ψ_M) will represent a RN-space, and $(\Upsilon, \kappa, \Psi_M)$ will represent a complete RN-spaces. Regarding the mapping $\aleph: \chi \to \Upsilon$, we establish

$$D(\alpha, \beta) = \aleph(2\alpha + \beta) + \aleph(2\alpha - \beta) - 2\aleph(\alpha + \beta) - 2\aleph(\alpha - \beta) - 12\aleph(\alpha)$$

for all $\alpha, \beta \in \chi$.

In this work, we study the generalized Hyers-Ulam stability of the Cubic functional equation (1) in random normed spaces under the minimal $\hbar$-norm, utilizing the direct and fixed-point methods.

2. Stability of Functional Equation in Random Normed Space:

In this part, we are using direct method to study the generalized Ulam-Hyers stability of the functional equation in the RN space.

Theorem 2 Let $\phi: \chi^2 \to Z$ be a function such that for some $0 < a < 2^3$,

$$\kappa'_{\phi(2\alpha,2\beta)}(\hbar) \geq \kappa'_{a\phi(\alpha,\beta)}(\hbar) \tag{2}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$ and

$$\lim_{n\to\infty} \kappa'_{\phi(2^n\alpha,2^n\beta)}(2^{3n}\hbar) = 1, \tag{3}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$. If $\aleph: \chi \to \Upsilon$ is a mapping with $\aleph(0) = 0$ such that

$$\kappa_{D\aleph(\alpha,\beta)}(\hbar) \geq \kappa'_{\phi(\alpha,\beta)}(\hbar) \tag{4}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$, then there exists a unique cubic mapping $Q: \chi \to \Upsilon$, such that

$$\kappa_{(\aleph(\alpha)-Q(\alpha))}(\hbar) \geq \kappa'_{\phi(\alpha,0)}\left(2(2^2 - a)\right) \tag{5}$$

for all $\alpha, \beta, z \in \chi$ and $\hbar > 0$.

Proof: Replacing (α, β) by $(\alpha, 0)$ in equation (4), we get

$$\kappa_{\frac{1}{2^3}\aleph(2\alpha)-\aleph(\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(16\hbar) \tag{6}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$.

Replacing α by $2^n\alpha$ in equation (6), we get

$$\kappa_{\left(\frac{\aleph(2^{n+1}\alpha)}{2^{3(n+1)}}-\frac{\aleph(2^n\alpha)}{2^{3n}}\right)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}\left(\left(\frac{2^3}{a}\right)^n (16\hbar)\right) \tag{7}$$

for all $\alpha \in \chi$ and $\hbar > 0$, since

$$\frac{\aleph(2^n)}{2^{3n}} - \aleph(\alpha) = \sum_{j=0}^{n-1} \left(\frac{\aleph(2^{j+1}\alpha)}{2^{3(j+1)}} - \frac{\aleph(2^j\alpha)}{2^{3j}}\right) \tag{8}$$

$$\kappa_{\left(\frac{\aleph(2^n\alpha)}{2^{3n}}-\aleph(\alpha)\right)}\left(\sum_{j=0}^{n-1}\left(\frac{a}{2^3}\right)^j \hbar\right) \geq \Psi_M\left(\left(\kappa'_{\phi(\alpha,0)}(\hbar)\right)\right)$$

$$= \kappa'_{\phi(\alpha,0)}(\hbar) \tag{9}$$

for all $\alpha \in \chi$ and $\hbar > 0$. Replacing α by $2^m\alpha$ in equation (9), we get

$$\kappa_{\left(\frac{\aleph(2^{n+m}\alpha)}{2^{3(n+m)}}-\frac{(2^m\alpha)}{2^{3m}}\right)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}\left(\frac{16\hbar}{\sum_{j=m}^{n+m-1}\left(\frac{a}{2^3}\right)^j}\right) \tag{10}$$

This implies that $\left(\frac{\aleph(2^n\alpha)}{2^{3n}}\right)$ is a cauchy sequence in complete RN space, so it converges to some point $Q(\alpha) \in \Upsilon$, for all $\alpha \in \chi$ and $\hbar > 0$. Letting $m = 0$ in equation (10) we get

$$\kappa_{\left(\frac{\aleph(2^n\alpha)}{2^{3n}}-\aleph(\alpha)\right)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}\left(\frac{16\hbar}{\sum_{j=0}^{n-1}\left(\frac{a}{2^3}\right)^j}\right) \tag{11}$$

Let $Q(\alpha) = \lim_{n\to\infty}\frac{\aleph(2^n\alpha)}{2^n}$, and for any $\delta > 0$ we have

$$\kappa_{Q(\alpha)-\aleph(\alpha)}(\delta+\hbar) \geq \Psi_M\left(\kappa_{Q(\alpha)-\frac{\aleph(2^n\alpha)}{2^{3n}}}(\delta), \kappa_{\frac{\aleph(2^n\alpha)}{2^{3n}}-\aleph(z)}(\hbar)\right)$$

$$\geq \Psi_M\left(\kappa_{Q(\alpha)-\frac{(2^n\alpha)}{2^{3n}}}(\delta), \kappa'_{\phi(\alpha,0)}\left(\frac{16\hbar}{\sum_{j=0}^{n-1}\left(\frac{a}{2^3}\right)^j}\right)\right) \tag{12}$$

for all $\alpha \in \chi$ and $\hbar > 0$. Letting $n \to \infty$, in equation (12), we get

$$\kappa_{Q(\alpha)-\aleph(\alpha)}(\delta+\hbar) \geq \kappa'_{\phi(\alpha,0)}(2(2^3-a)\hbar) \tag{13}$$

for all $\alpha \in \chi$ and $\hbar > 0$. Letting $\delta \to 0$, we obtain

$$\kappa_{Q(\alpha)-\aleph(\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(2(2^3-a)\hbar) \tag{14}$$

So, condition of equation (5) hold.

Replacing α by $2^n\alpha$ and β by $2^n\beta$ in equation (4) respectively, we get

$$\kappa_{\frac{D\aleph(2^n\alpha,2^n\beta)}{2^{3n}}}(\hbar) \geq \kappa'_{\phi(2^n\alpha,2^n\beta)}(\hbar 2^{3n}) \tag{15}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$. Letting $n \to \infty$, in equation (15), we get Q satisfy the equation (2).

To prove the uniqueness: let, if possible, there exists $H: \chi \to \Upsilon$ which satisfying equation (5). Hence

$$Q(2^n z) = 2^{2n} Q(z)$$

and

$$H(2^n z) = 2^{2n} H(z)$$

Thus

$$\kappa_{Q(\alpha)-H(\alpha)}(\hbar) = \kappa_{\frac{Q(2^n\alpha)}{2^{3n}}-\frac{H(2^n\alpha)}{2^{3n}}}(\hbar)$$

$$\geq \Psi_M(\kappa_{\left(\frac{Q(2^n\alpha)}{2^{3n}}-\frac{\aleph(2^n\alpha)}{2^{3n}}\right)}\left(\frac{\hbar}{2}\right), \kappa$$

$$\geq \kappa'_{\phi(\alpha,0)}\left(2(2^3-a)\left(\frac{2^3}{a}\right)^n \hbar\right) \tag{16}$$

for all $\alpha \in \chi$ and all $\hbar > 0$. Since,

$$\lim_{n\to\infty}\left(2(2^3-a)\left(\frac{2^3}{a}\right)^n \hbar\right) = \infty$$

we have

$$\kappa_{Q(\alpha)-H(\alpha)}(\hbar) = 1$$

for all $\hbar > 0$. thus, the Quadratic mapping Q is unique.

Theorem 3. Let $\phi: \chi^2 \to Z$ be a function such that, for some $2^3 < \alpha$,

$$\kappa'_{\phi\left(\frac{\alpha}{2},\frac{\beta}{2}\right)}(\hbar) \geq \kappa'_{\phi(\alpha,\beta)}(\alpha\hbar) \tag{17}$$

and $\lim_{n\to\infty}\kappa'_{2^{3n}\phi\left(\frac{\alpha}{2^n},\frac{\beta}{2^n}\right)}(\hbar) = 1$

for all $\alpha, \beta \in \chi$ and all $\hbar > 0$. If $\aleph: \chi \to \Upsilon$ is mapping such that $\aleph(0) = 0$ and satisfies equation (4), then there exists a unique quadratic mapping $Q: \chi \to \Upsilon$ such that

$$\kappa_{\aleph(\alpha)-Q(\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}\left(2(a-2^3)\right) \tag{18}$$

for all $\alpha \in \chi$.

Proof: It follows from equation (4) we get

$$\kappa_{\left(\aleph(\alpha)-2^3\aleph\left(\frac{\alpha}{2}\right)\right)} \geq \kappa'_{\phi(\alpha,0)}(2a\hbar) \tag{19}$$

for all $\alpha \in \chi$. Using the triangular inequality and equation (11), we get

$$\kappa_{\left(\aleph(\alpha)-2^{3n}\aleph\left(\frac{\alpha}{2^n}\right)\right)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}\left(\frac{2a\hbar}{\sum_{j=m}^{m+n-1}\left(\frac{2^3}{a}\right)^j}\right) \tag{20}$$

for all $\alpha \in \chi$ and $m, n \in Z$ with $n > m \geq 0$. Then the sequence $\left\{2^{3n}\aleph\left(\frac{z}{2^n}\right)\right\}$ is a Cauchy sequence in the complete RN-space, so it converges to some point $Q(\alpha) \in \Upsilon$.

We can define a mapping $Q: \chi \to \Upsilon$ by

$$Q(\alpha) = \lim_{n\to\infty} 2^{3n}\aleph\left(\frac{\alpha}{2^n}\right) \tag{21}$$

for all $\alpha \in \chi$. Then the mapping 1 satisfies the equation (1) and (20).The remaining proof is same as in theorem (1). this completes the proof.

Corollary 1: Let θ be a non negative real number and z_0 be a unique fixed point of Z. If $\aleph: \chi \to \Upsilon$ is a mapping with $\aleph(0) = 0$ which satisfies

$$\kappa_{D(\aleph(\alpha,\beta))}(\hbar) \geq \kappa'_{\theta z_0}(\hbar) \tag{22}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$, then there exists a unique quadratic mapping $C: \chi \to \Upsilon$ such that

$$\kappa_{\aleph(\alpha)-Q(\alpha)}(\hbar) \geq \kappa'_{\theta z_0}(14t) \tag{23}$$

for all $z \in \chi$ and $\hbar > 0$.

Proof: Let $\phi: \chi^2 \to Z$ be defined by $\phi(\alpha, \beta) = \theta z_0$. Then, by using theorem (1) for $a = 1$ we can obtain the desired result. This concludes the proof.

Corollary 2: Let us suppose that $p, q \in R$ be a positive real number with $p, q < 3$ and z_0 be a fixed unit point of Z. If $\aleph: \chi \to \Upsilon$ is a mapping with $\aleph(0) = 0$ which satisfies

$$\kappa_{D\aleph(\alpha,\beta)}(\hbar) \geq \kappa'_{(\|\alpha\|^p + \|\beta\|^q)z_0}(\hbar) \tag{24}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$, then there exists a unique cubic mapping $Q: \chi \to \Upsilon$ such that

$$\kappa_{\aleph(\alpha)-Q(\alpha)}(\hbar) \geq \kappa'_{\|\alpha\|^p z_0}(2(2^3 - 2^p)\hbar) \tag{25}$$

for all $\alpha \in \chi$ and $\hbar > 0$.

Proof: Let $\phi: \chi^2 \to Z$ be defined by $\phi(\alpha, \beta) = (\| \alpha \|^p + \| \beta \|^q)z_0$. Then the proof follow from theorem 1 by $a = 2^p$. this complete the proof.

Theorem 4. [9] Suppose that (Ω, s) is a complete generalized metric space and $J: \Omega \to \Omega$ is a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for each $\alpha \in \Omega$, either $s(J^n\alpha, J^{n+1}) = \infty$ for all non-negative integers $n \geq 0$ or there exists a natural number n_0 such that

(1) $s(J^n\alpha, J^{n+1}\alpha) < \infty$ for all $n \geq n_0$;

(2) The sequence $\{J^n\alpha\}$ is convergent to a fixed point β^* of J;

(3) β^* is the unique fixed point of J in the set $A = \{\beta \in \Omega: s(J^{n_0}\alpha, \beta) < \infty\}$;

(4) $s(\beta, \beta^*) \leq \frac{1}{1-L} s(\beta, J\beta)$ for all $\beta \in A$.

Theorem 5 Let $\phi: \chi^2 \to D^+$be a function such that, for some $0 < a < 2^3$,

$$\kappa'_{\phi(\alpha,\beta)}(\hbar) \leq \kappa'_{\phi(2\alpha,2\beta)}(a\hbar) \tag{26}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$.If $\aleph: \chi \to \Upsilon$ is a mapping with $\aleph(0) = 0$ such that

$$\kappa_{D\aleph(\alpha,\beta)}(\hbar) \geq \kappa'_{\phi(\alpha,\beta)}(\hbar) \tag{27}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$, then there exists a unique cubic mapping $Q: \chi \to \Upsilon$ such that

$$\kappa_{\aleph(\alpha)-Q(\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(2(2^3 - a)\hbar) \tag{28}$$

for all $\alpha \in \chi$ and $\hbar > 0$

Proof : Taking $\beta = 0$ in equation (27), we get

$$\kappa_{\aleph(\alpha)-\frac{1}{2^3}(2\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(16\hbar) \tag{29}$$

for all $\alpha \in \chi$ and $\hbar > 0$. Let $\Omega = \{g: \chi \to \Upsilon, g(\alpha) = 0\}$ and the mapping s defined on Ω by

$$s(g,h) = \inf\{r \in [0,\infty): \kappa_{g(\alpha)-h(\alpha)}(r\hbar) \geq \kappa'_{\phi(\alpha,0)}(\hbar),\} \tag{30}$$

for all $\alpha \in \chi$ Where, as usual $\inf\phi = -\infty$. then (Ω, s) is p generalized complete metric space. Now, let us consider the mapping $J: \Omega \to \Omega$ defined by

$$Jg(\alpha) = \frac{1}{2^3} g(2z)$$

for all $g \in \Omega$ and $\alpha \in \chi$. Let $g, h \in \Omega$ and $r \in [0,\infty)$ be an arbitrary constant

$s(g,h) < r$.Then

$$\kappa_{g(\alpha)-h(\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(\hbar) \tag{31}$$

for all $z \in \chi$ and $\hbar > 0$ and so

$$\kappa_{Jg(\alpha)-Jh(\alpha)}\left(\frac{ar\hbar}{2^3}\right) = \kappa_{g(2\alpha)-h(2\alpha)}(ar\hbar)$$

$$\geq \kappa'_{\phi(\alpha,0)}(\hbar) \tag{32}$$

for all $\alpha \in \chi$ and $\hbar > 0$. Hence, we have

$$s(Jg, Jh) \leq \frac{pr}{2^3} \leq \frac{a}{2^3} s(g,h) \tag{33}$$

for all $g, h \in \Omega$. Then J is a contractive mapping on Ω with Lipchitz constant $L = \frac{a}{2^3} < 1$.

Thus, it follows from Theorem 4 there exists a mapping $Q: \chi \to \Upsilon$, which is a unique fixed point of J in the set $\Omega_1 = \{g \in \Omega: s(g,h) < \infty\}$, such that

$$Q(\alpha) = \lim_{n\to\infty} \frac{\aleph(2^n\alpha)}{2^{3n}} \tag{34}$$

for all $\alpha \in \chi$ since $\lim_{n\to\infty} s(J^n\aleph, Q) = 0$ also from

$$\kappa_{\aleph(\alpha)-\aleph\left(\frac{2\alpha}{2^3}\right)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(16\hbar), \tag{35}$$

it follows that $s(\aleph, J\aleph) \leq \frac{1}{16}$ therefore using Theorem 4 again, we get

$$s(\aleph, Q) \leq \frac{1}{1-L} s(\aleph, J\aleph) \leq \frac{1}{2(2^3-a)} \tag{36}$$

This means that

$$\kappa_{\aleph(\alpha)-Q(\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(2(2^3-a)\hbar) \tag{37}$$

for all $\alpha \in \chi$ and $\hbar > 0$. Also, by replacing α by $2^n\alpha$ and β by $2^n\beta$ in equation (27), respectively, we get

$$\kappa_{DQ(\alpha,\beta)}(\hbar) \geq \lim_{n\to\infty} \kappa'_{\phi(2^n\alpha, 2^n\beta)}(2^n\hbar)$$

$$= \lim_{n\to\infty} \kappa'_{\phi(\alpha,\beta)}\left(\left(\frac{2^3}{a}\right)^n \hbar\right) = 1 \tag{38}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$. The mapping Q is quadratic when ($RN1$) is utilized. Assuming that there is a quadratic mapping $Q' : \chi \to \Upsilon$ that satisfies (28), we can demonstrate the uniqueness of the result. Thus Q is a fixed point of J in Ω_1. However, it follows from theorem 4 that J has only one fixed point in Ω_1. Hence $Q = Q'$. This complete the proof.

Theorem 6 Let $\phi: \chi^2 \to D^+$be a function such that, for some $0 < 2^3 < a$,

$$\kappa'_{\phi(\alpha,\beta)}(\hbar) \geq \kappa'_{\phi\left(\frac{\alpha}{2},\frac{\beta}{2}\right)}(a\hbar) \tag{39}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$. If $\aleph: \chi \to \Upsilon$ is a mapping with $\aleph(0) = 0$ which satisfies (28) then there exists a unique quadratic mapping $Q: \chi \to \Upsilon$ such that

$$\kappa_{\aleph(\alpha)-Q(\alpha)}(\hbar) \geq \kappa'_{\phi(\alpha,0)}(2(a-2^2)\hbar) \tag{40}$$

for all $\alpha \in \chi$ and $\hbar > 0$.

Proof: By a modification in the proof of Theorem 2 and 5 we can easily obtain the desired result. This concludes the proof.

Corollary 3. Let χ be a Banach space, ϵ and p be a positive real number with $p \neq 3$. Assume that $\aleph: \chi \to \Upsilon$ is p mapping with $\aleph(0) = 0$ which satisfies

$$\| D\aleph(\alpha, \beta) \| \leq \epsilon(\| \alpha \|^p + \| \beta \|^p) \tag{41}$$

for all $\alpha, \beta, z \in \chi$.Then there exists a unique quadratic mapping $Q: \chi \to \Upsilon$ such that

$$\| Q(\alpha) - \aleph(\alpha) \| \leq \frac{\epsilon \|\alpha\|^p}{2(|2^3 - 2^p|)} \tag{42}$$

for all $\alpha \in \chi$ and $\hbar > 0$.

Proof: Define $\kappa: \chi * \mathbb{R} \to \mathbb{R}$ by

$$\kappa_\alpha(\hbar) = \begin{cases} \frac{\hbar}{\hbar + ||\alpha||} & \hbar \leq 00 \quad \hbar > 0 \end{cases}$$

for all $\alpha \in \chi$ and $\hbar \in \mathbb{R}$. Then (χ, κ, Ψ_M) is a complete RN-space. Denote $\phi: \chi \times \chi \to R$ by

$$\phi(\alpha, \beta) = \epsilon(\| \alpha \|^p + \| \beta \|^p) \tag{43}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$. It follows from

$$\| D\aleph(\alpha, \beta) \| \leq \theta(\| \alpha \|^p + \| \beta \|^p) \tag{44}$$

That is

$$\kappa_{D\aleph(\alpha,\beta)}(\hbar) \geq \kappa'_{\phi(\alpha,\beta)}(\hbar) \tag{45}$$

for all $\alpha, \beta \in \chi$ and $\hbar > 0$, where $\kappa': \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\kappa'_\alpha(\hbar) = \begin{cases} \frac{\hbar}{\hbar + ||\alpha||} & \hbar \leq 00 \quad \hbar > 0 \end{cases}$$

is a random norm on $\mathbb{R}$, then all the condition of Theorem 5 and 6 hold and so there exists a unique quadratic mapping $Q: \chi \to \Upsilon$ such that

$$\frac{\hbar}{\hbar + \|Q(\alpha) - \aleph(\alpha)\|} = \kappa_{Q(\alpha) - \aleph(\alpha)}(\hbar)$$

$$\geq \kappa'_{\phi(\alpha,0)}(2(|2^3 - a|\hbar)$$

$$= \frac{2(|2^3 - a|\hbar)}{2(|2^3 - a|\hbar) + \epsilon\|\alpha\|^p} \tag{46}$$

Therefore we obtain the required result by taking $a = 2^p$.

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