A NEW TYPE OF PARTIAL b −METRIC SPACE AND FIXED POINT THEOREMS

Satyaj Tiwari [1] and Lokesh Kumar Dewangan [2]

[1,2]Department of Mathematics, Shri Rawatpura Sarkar University,

Raipur, 492015, Chhattisgarh

Email: [1]satyaj.tiwari@sruraipur.ac.in, [2]srilokesh94@gmail.com

Abstract

In this study, we extend the concept of partial b-metric space to introduce Generalized partial b-metric space, building upon the work by S. Shukla in [25]. This new framework is motivated by the properties of partial b-metric spaces. We provide examples that illustrate scenarios not meeting the criteria of a partial b-metric space. Furthermore, our research includes the development and proof of several fixed point theorems. Notably, our findings generalize and expand upon various established results from existing literature.

Keywords and phrases. Generalized partial b-metric space, Cauchy sequence, fixed point.

2020 Mathematical Sciences Classification. 47H10, 54H25.

1. Introduction

In recent decades, there has been significant interest in advancing fixed point theory due to its numerous applications within metric spaces [1, 2]. A pivotal tool in this context is Banach's fixed point theorem [3], which holds wide relevance across diverse mathematical areas [4, 5, 6, 7, 8, 9, 10, 11, 12]. Scholars have made a concerted effort in recent years to extend this theorem to various generalized metric spaces [13, 14, 15, 16, 17].

In subsequent developments, Bakhtin [18] and Czerwik [19] introduced b-metric spaces, an extension of metric spaces (also discussed in [20]). They established the contraction mapping principle for b-metric spaces, thus extending the well-known Banach contraction principle to this context. This laid the foundation for exploring fixed point theory within b-metric spaces, covering both single-valued and multi-valued operators.

In 1994, Matthews (as referenced in [21, 22]) introduced partial metric spaces as part of denotational semantics for data flow networks. In these spaces, conventional metrics are replaced by partial metrics, where the self-distance of any point is not required to be zero. Matthews also demonstrated the applicability of the Banach contraction principle [3] within partial metric spaces. Notably, partial metric spaces are vital in constructing models in computation theory (e.g., [23], [24], among others). Heckmann [23] later extended this concept by relaxing the small self-distance axiom, giving rise to the notion of weak partial metrics.

More recently, Shukla [25] expanded upon the ideas of b-metric spaces and partial metric spaces, introducing the concept of partial b-metric spaces. In this context, Shukla not only formulated the Banach contraction principle but also established a Kannan-type fixed point theorem within partial b-metric spaces, providing illustrative examples to support these novel findings.

In this paper, we introduce the concept of a generalized partial b-metric space and provide an example that demonstrates our definition. Additionally, we present several fixed point theorems applicable to generalized partial b-metric spaces.

2. Preliminaries

Firstly, we recall some basic definitions of metric spaces,

Definition 2.1 *[19]Let X be a non-empty set and the self mapping $d: X \times X \to R^+$ (R^+ stands for non-negative reals) satisfies*:

(bM1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(bM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(bM3) there exist a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$

Then d is called a b-metric on X and (X, d) is called a b −metric space with coefficient s.

Definition 2.2 *[21]Let X be a non-empty set and the self mapping $d: X \times X \to R^+$ (R^+ stands for non-negative reals) satisfies:*

(PM1) $x = y$ if and only if $d(x, x) = d(x, y) = d(y, y)$;

(PM2) $d(x, x) \leq d(x, y)$

(PM3) $d(x, y) = d(y, x)$

(PM3) $d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$

Then d is called a partial metric on X and (X, d) is called a partial metric space.

Definition 2.3 *[25]Let X be a non-empty set and the self mapping $d: X \times X \to R^+$ (R^+ stands for non-negative reals) satisfies*:

(Pb1) $x = y$ if and only if $d(x,x) = d(x,y) = d(y,y)$;

(Pb2) $d(x,x) \leq d(x,y)$;

(Pb3) $d(x,y) = d(y,x)$;

(Pb4) there exist a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,z) + d(z,y)] - d(z,z)$.

Then d is called a partial b −metric on X and (X,d) is called a partial b −metric space with coefficient s.

Remark 2.1*In a partial b −metric space (X,d) if $x, y \in X$ and $d(x,y) = 0$, then $x = y$, but the converse may not be true.*

Remark 2.2*It is clear that every partial metric space is a partial b −metric space with coefficient $s = 1$ and every b −metric space is a partial b −metric space with the same coefficient and zero distance. However, the converse of this fact need not hold.*

Example 2.1*Let $X = R^+$, $q > 1$ be a constant, and $d: X \times X \to R^+$ be defined by*

$$d(x,y) = [\max\{x,y\}]^q + |x-y|^q,$$

for all $x, y \in X$.

Then (X,d) is partial b −metric space with $s = 2^q > 1$ but it is neither a b −metric nor a partial metric space. Indeed, for $x > 0$ we have $d(x,x) = x^q \neq 0$; therefore, d is not a b −metric on X. Also for $x = 6, y = 2, z = 3$ we have $d(x,y) = 6^q + 4^q$ and $d(x,z) + d(z,y) - d(z,z) = 6^q + 4^q - 1^q$. So $d(x,y) > d(x,z) + d(z,y) - d(z,z)$. Hence d is not a partial metric on X but it is partial b −metric for $s = 2^q$.

Definition 2.4 Let $\{x_n\}$ be a sequence in a partial b −metric space (X,d). Then:

(1) The sequence $\{x_n\}$ is said to be a convergent in (X,d), if there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$.

(2) The sequence $\{x_n\}$ is said to be a Cauchy sequence in (X,d), if for every $\varepsilon > 0$ there exists a positive $n_0 \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0$ (or, equivalently, $\lim_{n,m\to\infty} d(x_n, x_m) = 0$).

(3) (X,d) is called a complete partial b −metric space if every Cauchy sequence is convergent in X.

3.Main Results

We introduce a Generalized partial b-metric space by the generalization of partial b-metric space is defined by:

Definition 3.1 *Let X be a non-empty set and the self mapping $d: X \times X \to R^+$ (R^+ stands for non-negative reals) satisfies*:

(Gp1) $x = y$ if and only if $d(x,x) = d(x,y) = d(y,y)$

(Gp2) $d(x,x) \leq d(x,y)$

(Gp3) $d(x,y) = d(y,x)$

(Gp4) there exist a real number $s \geq 1$ such that $d(x,y) \leq s[d(x,z) + d(z,w) + d(w,y)] - d(z,z) - d(w,w)$

Then d is called a generalized partial b-metric on X and (X,d) is called a generalized partial b-metric space with coefficient s.

Example 3.1 Let $X = \mathbb{R}^+$, and $0, \alpha \in \mathbb{R}$ then define $d: X \times X \to X$

$$\begin{cases} 0 & if x = y \\ 3\alpha if x, y \in \{1,2\} \\ \alpha otherwise \end{cases}$$

Then *(X,d)* is a generalized partial b-metric space but it is not a partial b-metric space. Indeed, $d(1,2) \leq s[d(1,3) + d(3,4) + d(4,2)] - d(3,3) - d(4,4)$, i.e., $3\alpha \leq 3s\alpha$ but $d(1,2) \nleq s[d(1,3) + d(3,2)] - d(3,3)$, i.e. $3\alpha \nleq 2\alpha s$.

Then it is easy to see that (X,d) is a generalized partial b-metric space but it is not a partial b −metric space.

Theorem 3.1 *Let (X,d) be a Housdorff and complete generalized partial b-metric space , and the mapping $T: X \to X$ satisfying the following contraction mapping*

$$d(Tx,Ty) \leq \lambda d(x,y)$$

for all x, y in X, where λ is called contractive constant of T. then T has a unique fixed point.

Proof. Let x is an element of X, $a_n = T^n x$ for $n \geq 0$ and $c = \inf S$, where

$$S = \{d(a_{n-1}, a_n): n \in N\}$$

Now, we claim that $c = 0$

But if $c \neq 0$ then $c < \frac{c}{\lambda}$ and hence there is a positive integer n such $d(a_{n-1}, a_n) < \frac{c}{\lambda}$ so that $\lambda d(a_{n-1}, a_n) < c$, By contractive property of T, we have

$$d(T^n x, T^{n+1} x) < c$$

a contradiction to the minimality of c, therefore $c = 0$. The monotonically decreasing property of the sequence $d(a_n, a_{n+1})$ implies that $d(a_n, a_{n+1})$ converges to 0 ...(2.1)

Again we claim that T has a periodic point. Suppose, to obtain a contradiction, T has no periodic point. Then $\{a_n\}$ is a sequence of distinct points for $m > n+1$, we have

$$d(a_n, a_m) = d(T^n x, T^m x)$$

$$\leq s[d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{m+1} x) + d(T^{m+1} x, T^m x)]$$

$$-d(T^{n+1} x, T^{n+1} x) - d(T^{m+1} x, T^{m+1} x)$$

$$\leq s[\lambda^n d(x, Tx) + \lambda d(T^n x, T^m x) + \lambda^m d(Tx, x)]$$

$$\leq s(\lambda^n + \lambda^m) d(x, Tx) + s\lambda d(T^n x, T^m x)$$

$$(1 - s\lambda) d(a_n, a_m) \leq s(\lambda^n + \lambda^m) d(x, Tx)$$

which implies that $\{a_n\}$ is a Cauchy sequence in (X,d)(by equation (2.1)). By completeness, $a_n \to a$ for some a in X. Also $d(Ta_n, Ta) \leq \lambda d(a_n, a)$ and $d(a_n, a) \to 0$. So $d(Ta_n, Ta) = d(a_{n+1}, Ta) \to 0$. Hence $a_n \to a$ and $a_{n+1} \to Ta$. Since (X, d) is Housdorff it follows that $a = Ta$, a contradiction to the assumption that T has no periodic point. Thus T has a periodic point say a of period n. Suppose if possible $n > 1$. Then $d(a, Ta) = d(T^n a, T^{n+1} a) < \lambda^n d(a, Ta)$, a contradiction. So $n = 1$ and a is a fixed point of T. If a, b are fixed points of T then $d(a, b) = d(Ta, Tb) \leq d(a, b)$. Since $0 < \lambda < 1$, We have $a = b$

Following theorem is an analog to Kannan fixed point theorem see [26] in generalized partial b-metric space:

Theorem 3.2 *Let (X, d) be a complete generalized partial b-metric space, and the mapping $T: X \to X$ satisfied Kannan contraction, then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X. Let $x_1 = T(x_0)$. If $x_1 = x_0$ then $T(x_0) = x_0$ this means x_0 is fixed point of T and there is nothing to prove.

Assume that $x_1 \neq x_0$ Let $x_2 = T(x_1)$. In this way we can define a sequence of points in X as follows :

$$x_{n+1} = T(x_n) = T^{n+1}(x_0) \ \forall \, n = 1,2,3, \ldots$$

using the inequality (1.1), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

$$\leq \lambda\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}$$

$$\leq \lambda\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}$$

$$\leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n)$$

we can also suppose that x_0 is not periodic point, In fact if $x_n = x_0$, then

$$d(x_0, Tx_0) = d(x_n, Tx_n)$$

$$\leq d(T^n x_0, T^{n+1} x_0)$$

$$\leq (\frac{\lambda}{1-\lambda}) d(T^{n-1} x_0, T^n x_0)$$

$$\leq (\frac{\lambda}{1-\lambda})^2 d(T^{n-2} x_0, T^{n-1} x_0)$$

$$\ldots$$

$$\ldots$$

$$\leq (\frac{\lambda}{1-\lambda})^n d(x_0, Tx_0)$$

put $h = \frac{\lambda}{1-\lambda}$ then $h \leq 1$ and $(1 - h^n) d(x_0, Tx_0) \leq 0$ this implies $d(x_0, Tx_0) \leq 0$

It follows that x_0 is a fixed point of T. Thus in the sequal of proof we can suppose

$$T^n x_0 \neq x_0$$

Now inequality (1) implies that

$$d(T_0^x, T^{n+m}x_0) \leq \lambda\{d(T^{n-1}x_0, T^n x_0) + d(T^{n+m-1}x_0, T^{n+m}x_0)\}$$

$$\leq \lambda\{h^{n-1}d(x_0, Tx_0) + h^{n+m-1}d(x_0, Tx_0)\}$$

therefore, $d(x_n, x_{n+m-1}) \to 0$ as $n \to \infty$.

It implies that $\{x_n\}$ is a cauchy sequence in X. Since X is complete, there exist $u \in X$, such that $x_n \to u$

By 4th property of generalized partial b-metric space, we have

$$d(Tu, u) \leq s[d(Tu, T^n x_0) + d(T^n x_0, T^{n+1}x_0) + d(T^{n+1}x_0, u)]$$

$$-d(T^n x_0, T^n x_0) - d(T^{n+1}x_0, T^{n+1}x_0)$$

$$\leq s[d(Tu, T^n x_0) + d(T^n x_0, T^{n+1}x_0) + d(T^{n+1}x_0, u)]$$

$$\leq sd(Tu, T^n x_0) + sd(T^n x_0, T^{n+1}x_0) + sd(T^{n+1}x_0, u)$$

$$\leq s\{\lambda[d(u, Tu) + d(T^{n-1}x_0, T^n x_0)]\} + sd(T^n x_0, T^{n+1}x_0) + sd(T^{n+1}x_0, u)$$

$$d(Tu, u) \leq s\lambda d(u, Tu) + s\lambda d(T^{n-1}x_0, T^n x_0) + sd(T^n x_0, T^{n+1}x_0) + sd(T^{n+1}x_0, u)$$

$$(1 - s\lambda)d(u, Tu) \leq s\lambda h^{n-1}d(x_0, Tx_0) + sh^n d(x_0, Tx_0) + sd(T^{n+1}x_0, u)$$

$$d(Tu, u) \leq \frac{s\lambda}{(1 - s\lambda)}h^{n-1}d(x_0, Tx_0) + \frac{s}{(1 - s\lambda)}h^n d(x_0, Tx_0) + \frac{s}{(1 - s\lambda)}d(T^{n+1}x_0, u)$$

$$d(Tu, u) \leq \frac{s\lambda}{(1 - s\lambda)}h^{n-1}d(x_0, Tx_0) + \frac{s}{(1 - s\lambda)}h^n d(x_0, Tx_0) + \frac{s}{(1 - s\lambda)}d(Tx_n, u)$$

Letting $n \to \infty$ and using the fact that, $d(a_n, y) \to d(a, y)$ and $d(x, a_n) \to d(x, a)$.

Whenever a_n is a sequence in X with $a_n \to a \in X$, We have

$$d(Tu, u) \leq 0 + 0 + \frac{s}{(1-s\lambda)}d(Tu, u)$$

$$d(Tu, u) \leq 0.$$

This implies that $Tu = u$.

Now we have to show that T has a unique fixed point.

For this, assume that there exist another fixed point v in X, such that $Tv = v$.

Now,

$$d(v, u) = d(Tv, Tu)$$

$$d(v,u) \leq \lambda[d(v,Tv) + d(u,Tu)]$$

$$d(v,u) \leq \lambda[d(v,v) + d(u,u)]$$

$$d(v,u) \leq \lambda[d(v,u) + d(u,v)]$$

$$(1-2\lambda)d(v,u) \leq 0$$

$$d(v,u) \leq 0.$$

Hence $u = v$.

4.Conclusion

A partial b-metric space was introduced by S. Shukla in [25] and also give some fixed point theorem. Now we introduce Generalized partial b-metric space motivated by partial b-metric space and we give some example which is not a partial b-metric space and we also prove some fixed point theorems. Many known results in the literature are also generalized by our finding.

References

[1] A.C.M. Ran and M.C.B. Reuring, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Am. Math. Soc., **132** (2004), 1435-1443.

[2] H.Q. Wang, X.H. Song, L.L. Chen and W. Xie, A secret sharing scheme for quantum gray and color image based on encryption, Int. J. Theor. Phys., **58** (2019), 1626-1650.

[3] S. Banach, Sur operation dans les ensembles abstraits et leur application aux equation integral, Fund. Math., **3** (1922), 133-181.

[4] L. Chen, L.Gao and D. Chen, Fixed point theorems of mean non expansive set-valued mappings in Banach spaces, J. Fixed Point theory Appl., **19** (2017), 2129-2143.

[5] L. Chen, L. Gao and Y. Zhao, A new iterative scheme for finding attractive points of (α, β) −generalized hybrid set-valued mappings, J. Nonlinear Sci. Appl., **10** (2017), 1228-1237.

[6] L. Chen, J. Zou, Y. Zhao and M. Zhang, Iterative approximation of common attractive points of (α, β) −generalized hybrid set-valued mappings, J. Fixed Point theory Appl., **21** (2019), 58.

[7] D. Mihet, Banach contraction theorem in fuzzy metric spaces, Fuzzy Sets Syst., **144** (2004), 431-439.

[8] M. Ruzhansky, Y.J. Cho, P. Agrawal and I. Area, Adavances in Real and Complex Analysis with Applications, Springer: London, UK, (2017).

[9] K. Sawangsup, W. Sintunavarat and Y.J. Cho, Fixed point theorems for orthogonal F −contraction mappings on O-complete metric spaces, J. Fixed Point Theory Appl., **22** (2020), 10.

[10] P. Saipara, P. Kumam and Y.J. Cho, Random fixed point theorems for Hardy-Rodgers self-random operators with applications to random integral equations, Stochastics, **90** (2018), 297-311.

[11] S. Reich, Fixed points for contractive functions, Boll. Della Unione Mat. Ital., **5** (1972), 26-42.

[12] S. Reich, Kannan's fixed point theorem, Boll. Un. Math. Ital., **4** (1971), 1-11.

[13] C. Mongkolkeha, Y.J. Cho and P. Kumam, Fixed point theorems for simulation functions in b −metric spaces via the wt-distance, Appl. Gen. Topol., **18** (2017), 91-105.

[14] Y.J. Cho, T.M. Rassias and R. Saadati, Fuzzy Operator Theory in Mathematical Analysis, Springer: London, UK, (2018).

[15] A. Aghajani, M. Abbas and J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b −metric spaces, Math. Slovaca, **64** (2014), 941-960.

[16] S. Reich, Nonlinear evolution equations and nonlinear ergodic theorems, Nonlinear-Anal. Theory Methods Appl., **1** (1977), 319-330.

[17] S. Reich and A.J. Zaslavski, Genericity in Nonlinear Analysis, Springer: London, UK, (2014).

[18] I. A. Bakhtin, The contraction mapping principle in almost metric space, Funct. Anal. Gos. Ped. Inst. Unianowsk, **30** (1989), 26-37.

[19] S. Czerwik, Contraction mappings in b −metric spaces, Acta Mathematica Et Informatica, **1** (1993), 5-11.

[20] S. Czerwik, Nonlinear set-valued contraction mappings in b −metric spaces, Atti. Semin. Mat. Fis. Univ. Modena, **46** (1998), 263-276.

[21] S. G. Matthews, Partial metric topology, Research report 2012, Dept. Computer Science, University of Warwik, (1992).

[22] S. G. Matthews, Partial metric topology, Proceeding of 8th summer conference on topology and its applications, Annals of the New York Academy of Science, **728** (1994), 183-197.

[23] R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures, **7** (1999), 71-83.

[24] J. O. Olaleru, G. A. Okeke and H. Akewe, Coupled fixed point theorems for generalized ψ-mappings satisfying contractive condition of integral type on cone metric spaces, Int. J. Math. Model. Comput., **2(2)** (2012), 87-98.

[25] S. Shukla, Partial b −metric spaces and fixed point theorems, Mediter. J. Math., **11** (2014), 703-711.

[26] Kannan R., Some results on fixed points, Bull. Calcutta Math. Soc., **25**(1972), 727-730,.