Automorphism Groups of Rings

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ABSTRACT :

Automorphism group of ring focusing on its structure properties relation with ring's properties. A characterization of Automorphism subgroup of the symmetric group on set of unit R normal subgroup and quotient group of Automorphism.

Keywords : Automorphism group, ring, symmetric group, unit group outer automorphism group.

INTRODUCTION :

A group G acts as automorphisms on a ring R with no nilpotent elements, then there exist non-trivial fixed elements. It would be useful, however, to be able to construct fixed elements, such as traces, or other trace-like elements. A first step in this direction is a theorem of G. If a cyclic p-group acts faithfully on a ring with no nilpotent elements of characteristic $p > 0$, then the trace is non-trivial. In this paper, we investigate further when the trace is non-trivial, and more generally when there exist non-trivial "partial traces" in the fixed ring.

We first introduce a new construction, which we shall call the normal closure, for any semi-prime ring R. It is constructed as a subring of a certain quotient ring, taken with respect to two-sided ideals. The advantage of the normal closure is that it is much more closely tied to R than the quotient ring is; in particular, if R has no nilpotent elements, neither does the normal closure.

The relationship to automorphisms quotient ring to define certain "inner" and "outer" automorphisms. Some of his results on automorphisms of rings with no nilpotent elements can be simplified using the normal closure. As an example, we give a proof of his result that if G is a finite group of inner automorphisms of a ring R with no nilpotent elements, of characteristic $p \neq 0$,, then the commutator subgroup [G, G] contains no elements of order p. As a consequence, any finite simple group acting non-trivially on a domain *D* has non-trivial trace. Moreover, for any finite group *G* acting on *D*, there exist non-trivial "partial traces" in the fixed ring.

In a different direction, we use the normal closure to study the skew group ring *R* G* of a finite group *G* of automorphisms of a domain *R*. We show that *R*G* is semi-prime if and only if the trace is non-trivial, and that *R* G* is prime if and only if the elements of *G* are *R*-independent.

PRELIMINARY NOTATION AND DEFINITIONS :

Let *R* be semi- prime, and let f denote the filter of all ideals of *R* with zero annihilator. Let R_f be the ring of (left) quotients of *R* with respect to f; *i.e.*, $R_f = lim_{I \in f}$ $\mathrm{Hom}_R({}_R I, R)$. This was defined for prime rings and extended to semi-prime rings. The elements of R_f are equivalence classes of left *R*- module homomorphisms from some $I \in f$ into *R*. For each x∈ R, let $I_x \in f$ be the ideal associated with x, so that (0)≠ $I_x x \subseteq R$, *if* $x \neq 0$.

Now *R* may be imbedded in R_f via right multiplication on *R*. R_f is semi- prime, and the center C of R_f is a regular ring. When *R* is prime, R_f is also be the prime and C is a field.

We next consider automorphisms of *R*. If *g* is any automorphism of *R*, and $r \in R$, we denote by r^g the image of *r* under *g*. Let *g* has a unique extension to *R*. Define

$$\phi_g = \{x \in R_f \mid xr^g = rx, \text{ for all } r \in R\}$$

It can be shown that if $x \in \phi_g$,then $xr^g = rx$ for all $r \in R_f$..

DEFINITION : If *G* is a group of automorphisms of *R*, let $G_{inn} = \{g \in G \mid \phi_g \neq (0)\}$.

DEFINITION : If G is a group of automorphisms of R, we say G is X-outer if G_{inn} (e) and X-inner if $G = G_{inn}$.

We note that in other work the terms f-outer and f-inner were used. The present notation reflects the fact that the definition of G_{inn}.

Although G_{inn} will not always be a subgroup of G, it is closed under inverses and under conjugation by elements of G. However, G_{inn} is always a (normal) subgroup of G when R is prime. In this case, non-zero elements of ϕ_g, are invertible in R_f, and so G_{inn} consists of those elements of G which become inner when extended to R_f.

The algebra of the group is $B = \sum_{g \in G} \phi_g$.

The fixed ring of G on R is $R^G = \{x \in R \mid x^g = x, for all\ g \in G\}$.

When G is a finite group, the trace of an element x∈ R is $t_G(x) = \sum_{g \in G} x^g$, or simply $t(x)$ if there is no ambiguity about the group. If Λ is any non-empty subset of G, we define $t_\Lambda(x) = \sum_{g \in \Lambda} x^g$. We say that $t_\Lambda(x)$ is a partial trace function if $t_\Lambda(R) \in R^G$, and t_Λ, is non-trivial if $t_\Lambda(x) \neq 0$, some $x \in$ R.

The order of G will be denoted by $|G|$.

For any group of automorphisms of R, the skew group ring $R*G$ is defined to be $R*G = \Sigma_{g \in G} \oplus Rg$, with addition given component-wise and multiplication given as follows: if r, $s \in R$ and g, $h \in G$, then $(rg) \cdot (sh) = rs^{g^{-1}}$ gh∈Rgh. This definition may be contrasted with that of a twisted group algebra $C^t[G]$. In this construction, C is a commutative ring and $\alpha: G \times G \rightarrow C$ is a factor set. Then

$C^t[G]=\{\Sigma\oplus\ Cg \mid g\in\ G\}$; again, addition is component-wise but now multiplication of group elements is twisted; if $a, b\in C$ and $g, h\in G$, then $(a\overline{g}).(b\overline{h})=ab\alpha(g,h)\overline{gh}$.

Both of these constructions are special cases of the *crossed product*, in which there is both a group action on *R* and a twist of the multiplication of the group elements. When *R* is a field, this is just the classical crossed product; the construction for prime rings.

1. THE NORMAL CLOSURE OF A SEMI-PRIME RING :

We first observe that when *R* has no nilpotent elements, R_f does not necessarily inherit the same property. This may be seen from the following example. Let $R= F(x, y\}$ be the free algebra in two indeterminates over a field *F*. Then $I = Rx + Ry$ is an ideal of *R*. Define $f_1 : I \to R$ by $f_1(r_1x+r_2y)=r_1x$, and $f_2 : I \to R$ by $f_2(r_1x+r_2y)=r_2y$. Then f_1 and f_2 are left *R*-module mappings, so determine elements $f_1, f_2 \in R_f$. But $f_1.f_2 = 0$, even though *R* is a domain. Moreover, as R_f is prime, $f_2R_f\, f_1 \neq 0$, and this set consists of nilpotent elements. In this section we consider a subring of R_f, containing *R*, which is better behaved.

Throughout, *R* is semi-prime. Let $N = \{n \in R_f \mid nR = Rn\}$; that is, *N* is the set of *R*-normalizing elements of R_f. For *g* any automorphism of F, any element of ϕ_g, is *R*-normalizing and so $\phi_g \subseteq N$. In particular, $\phi_e = C \subseteq N$.

DEFINITION. The *normal closure* of *R* in R_f is the set

$$RN = \left\{\sum_i r_i n_i \in R_f \mid r_i \in R \cup \{1\}, n_i \in N\right\}$$

RN is clearly a subring of R_f, and is semi-prime since any non-zero ideal of *RN* intersects *R* non-trivially. Also, if *R* is prime, *RN* will be prime. Finally, the center of *RN* is just *C*, the extended centroid of *R*.

The crucial property of *RN* is the following:

PROPOSITION 1. *Let R be semi-prime. For any* $0\neq x\in RN$*, there exists* $I\in F$ *so that* $0\neq Ix\subseteq R$ *and* $0\neq xI\subseteq R$.

Proof. We first consider $n\in N$. By properties of R_F, there exists $J\in F$ such that $0\neq Jn\subseteq R$. We let $J'=\{a\in R \mid na\in Jn\}$. J' is an ideal of *R*, and if $na=0$, $a\in J'$. We claim that $J'\in F$. For if $J'b=0$, then $nJ'b=0=Jnb$. Since $J\in F$, $nb=0$. But then $b\in J'$. Since J' itself is a semi-prime ring, $J'b=0$ implies that $b=0$. Thus $J'\in F$.

Now choose $x\in RN$ and write $x=\sum_{i=1}^{k} r_i n_i$. By the above argument, for each n_i there exists $J_i\in F$ such that $n_i J_i\subseteq R$. Let $J=\bigcap_i J_i$; then $J\in F$, and $xJ\subseteq\sum_i r_i(n_i J_i)\subseteq R$. On the other hand, by the property of R_F, there exists $K\in F$ so that $0\neq Kx\subseteq R$. Let $I=J\cap K\in F$. Then $0\neq Ix\subseteq R$, and we claim that $0\neq xI\subseteq R$. Clearly $xI\subseteq R$, and if $x1=(0)$, we have $(0)=I(xI)=(Ix)I$, which is a contradiction since $I\in F$ and $Ix\neq(0)$. Thus $0\neq xI\subseteq R$.

COROLLARY 2. *(1) If R has no nilpotent elements, then RN has no nilpotent elements. (2) If R is a domain, then RN is a domain.*

Proof. (1) We follow an argument of Martindale for the central closure *RC*. Let $x\in RN$ with $x\neq 0$, $x^2=0$. There exists $I\in F$ so $0\neq xI\subseteq R$, $0\neq Ix\subseteq R$. Thus $0\neq xI^2x\subseteq R$. But xI^2x consists of nilpotent elements, so $xI^2x=(0)$. But then I^2x consists of nilpotent elements and $I^3x\subseteq R$, so $I^2x=(0)=I(Ix)$. Since $I\in F$, it follows that $Ix=(0)$, a contradiction. Thus $x=0$, and *RN* has no nilpotent elements.

(2) As *R* is prime, *RN* is prime, and has no nilpotent elements. Thus *RN* is a domain.

We note that when *R* is prime, *N**, the non-zero elements of *N*, consists precisely of those units $u \in R_F$ such that $uRu^{-1} = R.$ Each such unit therefore determines an *X*-inner automorphism of *R*.

Examples of the normal closure-

(1) Let *R* be a prime Goldie ring with classical ring of quotients *Q(R)*. Then *N** can be identified with those units $u \in Q(R)$ such that $uRu^{-1} = R$, and *RN* is simply the subring of *Q(R)* generated by *R* and *N*.

(2) Let *R* be a primitive ring with a minimal one-sided ideal. Let *e* be a primitive idempotent in *R*, so that *D* = *eRe* is a division ring. Then *V*= *eR* is a left vector space over *D*, *W* = *Re* is a right vector space, and *V* and *W* are dual spaces with a non-degenerate form <, >: $V \times W \to D$. Let $L = \{T \in Hom_D(V, V) / T$ has an adjoint relative to <, >}, the continuous *D*-linear transformations of *V*. By a well-known theorem of Jacobson, $R \in L$ and the socle *S* of *R* consists of all the finite rank transformations in *L*. Now, *N** can be identified with those units $u \in L$ such that $uRu^{-1} = R$; thus *RN* is simply the subring of *L* generated by *R* and *N*. In the special case when *R*= *S*, *N* consists of the set *U* of all units of *L*. Then *RN* consists of the subring of *L* generated by *U*, as the finite rank transformations are in the subring generated by *U*.

2. SKEW GROUP RINGS OVER DOMAINS

Let *R* be a prime ring. In the special case when *G* is a group of *X*-inner automorphisms, we may associate with the skew group ring *R*G* a twisted group algebra of *G* over the extended center *C* of *R*. The construction goes as follows: for each $g \in G$, choose $0 = x_g \in \phi_g$, and define $\bar{g} = x_{g-1} \in R_F * G.$ Then $\bar{g}.\bar{h} = \alpha(g,h)gh,$, where $\alpha(g,h) = x_g^{-1} x_{h-1}^{g-1} x_{(gh)-1}^{-1} \in \phi_e = C.$ Since $\alpha : G \times G \to C$ is a factor set, we may form the twisted group algebra $C^t[G]$ with respect to α. Since $C^t[G] = \{\sum a_g \bar{g} \mid a_g \in C, g \in G\} = \{\sum a_g x_{g-1} g\} \subseteq R_F * G, C^t[G]$ is actually a subring of $R_F * G$. The semiprimeness (or prime- ness) of $R*G$ may be reduced to that of $C^t[G]$, by the following:

PROPOSITION 3. *Let R be prime, and G a group of X-inner automorphisms of R. Let $C^t[G]$ be as above. Then R* G is prime or semi-prime, respectively, if and only if $C^t[G]$ is prime or semi-prime, respectively.*

THEOREM 4. *Let R*G be the skew group ring of the finite group G over the prime ring R. Then*

(1) If R has characteristic 0, then R G is semi-prime.*

*(2) If Rhas characteristic p>0, then R *G is semi-prime if and only if R*P (or equivalently $C^t[P]$) is semi-prime for all elementary abelian p-subgroups P of G_{inn}.*

We shall also need the following lemma, which was observed independently by *D*.

LEMMA 5. *Let R* G be the skew group ring of a group G of X-inner auto- morphisms of the prime ring R. Let $C^t[G]$ be as above, and B be the algebra of the group. Then there is a natural epimorphism*

$$C^t[G] \rightarrow B^{op}$$

Proof. Let o denote multiplication in B^{op}, the opposite ring of *B*. We may view $C^t[G]$ as the set of all $\sum a_g - 1g$,, where $a_{g-1} \in \phi_{g-1}$. Note that each $a_{g-1}g$ centralizes R_F. Define

$$\lambda: C^t[G] \rightarrow B^{op}$$

by $\lambda\left(\sum a_{g-1}g\right)=\sum a_{g-1}$. This mapping is clearly additive and onto. To see that it is a homomorphism, it suffices to consider elements of the form $\alpha = a_g - 1g$ and $\beta = b_{h-1}h$. Now $\alpha\beta = (a_{g-1}g)(b_{h-1}h) = b_{h-1}a_{g-1}gh$, and thus $\lambda(\alpha\beta) = b_{h-1}a_{g-1} = a_{g-1}\circ b_{h-1} = \lambda(\alpha)\circ\lambda(\beta)$.

We first consider the problem of when R* G is prime. When *R* is a field and | *G* | = n, then it is well-known that $R{*}G \cong M_n(R^G)$, the n × n matrices over the fixed field R^G, since *R*G* is just the crossed product with trivial factor set. This fact can be proved directly by using Dedekind's theorem that distinct automorphisms of a field are linearly independent.

DEFINITION. A set $\{g_\alpha\}$ of automorphisms of *R* is said to be *R-independent* if whenever $\sum_{i=1}^{n} r_i x^{gi} = 0,$ for some finite set $\{g_1,\ldots\ldots,g_n\},$ some $r_1,\ldots\ldots r_n \in R,$ and for all $x \in R$, then $r_i = 0,\ i = 1,\ldots\ldots n.$

Note that the elements of a group *G* being *R*-independent is equivalent to saying that *R* is a faithful *R* G* module, under the action $\sum r_g g.r = \sum r_g r^g$..

LEMMA 6. *Let G be a finite group of automorphisms of R such that R*G is prime. Then the elements of G are R-independent.*

Proof. Let $f = \sum_{g\in G} g,$ the formal sum. For any $r \in R$, $fr \in R{*}G$, and $gfr = fr$ for any $g \in G$. Say that $\sum_{g\in G} r_g x^g = 0$, for all $x \in R$, and consider $w = \sum r_g g^{-1} \in R * G$. Choose $x \in R$; then $wxfr = \sum_{g\in G} r_g g^{-1} xfr = \sum_{g\in G} r_g x^g g^{-1} fr = \sum r_g x^g fr = 0$. Thus since *(R*G)f=Rf*, it follows that *w(R*G) fr = (0)*, which contradicts R*G being prime, unless *w=0*. But then each $r_g = 0$.

THEOREM 7. *Let G be a finite group of automorphisms of a domain R. Then the following are equivalent:*

*(1) R*G is prime*

(2) $dim_C\ B = |G_{inn}|$ *, where B is the algebra of the group*

(3) the elements of G are R-independent

Proof. Let the $B^{op} \cong C^t[G_{inn}]$. Thus $C^t[G_{inn}]$ is a domain, since $B \subseteq RN$ is a domain by Corollary 2. But then R* G_{inn} is prime, by Proposition 3, and so R*G is prime.

Say that $\dim_C B < |G_{inn}|$. Then for some $g_1,...,g_n \in G_{inn}$ and $0 \neq a_i \in \phi_{gi}, i = 1,......,n$, we have $\sum_{i=1}^{n} a_i = 0$. Choose any $x \in R$; then $0 = x\left(\sum a_i\right) = \sum xa_i = \sum a_i x^{gi}$. Since $a_i \in R_F$, for all *i*, there exists $I \in F$ so that $(0) \neq Ia_i \subseteq R$, all *i= 1,..., n*. Choose $a \in I$ with $aa_1 \neq 0$;; then $\sum (aa_i) x^{gi} = 0$ for all *x*, which contradicts the *R*-independence of $\{g_1, ..., g_n\}$.

References :

1. S. A. AMITSUR, On rings of quotients, in "Symposia Mathematica" 8, pp. 149-164, Academic Press, New York/London, 1972.
2. G. BERGMAN AND I. M. ISAACS, Rings with fixed-point-free group actions, Proc. London Math. Soc. 2 (1973), 69-87.
3. J. FISHER AND S. MONTGOMERY, Semi-prime skew group rings, J. Algebra 52 (1978), 241-247.
4. I. N. HERSTEIN, "Non-Commutative Rings," Carus Mathematical Monographs No. 15, Math. Assoc. Amer., Washington, D.C., 1968.
5. V.K. Kharchenko, Generalized identities with automorphisms, Algebra i Logika 17 (1975),215-237 (*Algebra and Logic* (March 1976), 132-148)

6. S. Montgomery, Outer automorphisms of semi- prime rings, J. London Math. Soc. 18 (1978), 209-220.
7. S. Montgomery and D.S. Passman, Crossed products over prime rings, Israel J. Math. 31 (1978), 224-256.

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